

TEAM INVOLVED IN DEVELOPMENT OF LEARNING TEXTBOOK (APPLIED MATHEMATICS II)

POLYTECHNIC FACULTY

1. Sh. Satyavan Dhaka, HoD, Govt. Polytechnic, Umri
2. Mrs. Kiran, Senior Lecturer Mathematics, Govt. Polytechnic, Sonapat
3. Mrs. Sapna Sang, Lecturer Mathematics, SJPP, Damla
4. Sh. Ravi Bansal, Lecturer Mathematics, Govt. Polytechnic, Manesar

INDEX

UNIT NO.	TITLE	PAGE NO.
	SYLLABUS	i-iii
1.	DIFFERENTIAL CALCULUS	4-15
2.	DIFFERENTIAL CALCULUS AND ITS APPLICATIONS	16-33
3.	INTEGRAL CACULUS	34-56
4.	APPLICATION OF INTEGRATION, NUMERICAL INTEGRATION AND DIFFERENTIAL EQUATIONS	57-72
5.	STATISTICS AND SOFTWARE	73-96

SYLLABUS

APPLIED MATHEMATICS II (220022)

L	P
4	-

RATIONALE

Applied mathematics forms the backbone of engineering students. Basic elements of Differential calculus, Integral calculus and Differential Equations have been included in this course. This will develop analytical abilities to apply in engineering field and will provide continuing educational base to the students.

COURSE OUTCOMES

After undergoing the subject, the students will be able to:

CO1: Formulate the engineering problems into mathematical format with the use of differential equations and differential

CO2: Use the differentiation and Integration in solving various Mathematical and Engineering problems.

CO3: Calculate the approximate area under a curve by applying integration and numerical methods.

CO4: Understand the purposes of measures of central tendency and calculate the measures of central tendency (mode, median, mean) for a set of data.

CO5: Learn about basic fundamentals about MATLAB/ SciLab and mathematical calculation with MATLAB/ SciLab software.

DETAILED CONTENTS

UNIT I

Differential Calculus

- 1.1 Definition of function; Concept of limits (Introduction only) and problems related to four standard limits only.
- 1.2 Differentiation of x^n , $\sin x$, $\cos x$, e^x by first principle.
- 1.3 Differentiation of sum, product and quotient of functions.

UNIT II

Differential Calculus and Its Applications

- 2.1 Differentiation of trigonometric functions, inverse trigonometric functions. Logarithmic differentiation, successive differentiation (upto 2nd order)
- 2.2 Application of differential calculus in:
 - (a) Rate measures
 - (b) Maxima and minima

UNIT III

Integral Calculus

- 3.1 Integration as inverse operation of differentiation with simple examples.
- 3.2 Simple standard integrals and related problems, Integration by Substitution method and Integration by parts.
- 3.3 Evaluation of definite integrals with given limits.
Evaluation of $\int_0^{\pi/2} \sin^n x \, dx$, $\int_0^{\pi/2} \cos^n x \, dx$, $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$
using formulae without proof (m and n being positive integers only) using pre-existing mathematical models.

UNIT IV

Application of Integration, Numerical Integration and Differential Equations

- 4.1 Applications of integration: for evaluation of area under a curve and axes (Simple problems).
- 4.2 Numerical integration by Trapezoidal Rule and Simpson's $1/3^{\text{rd}}$ Rule using pre-existing mathematical models.

Differential Equations

- 4.3 Definition, order, degree, Type of differential Equations, linearity, Formulation of ordinary differential equation (up to 1^{st} order), solution of ODE (1^{st} order) by variable separation method.

UNIT V

Statistics and Software

Statistics

- 5.1 Measures of Central Tendency: Mean, Median, Mode
- 5.2 Measures of Dispersion: Mean deviation, Standard deviation

Software

- 5.3 SciLab software – Theoretical Introduction.
- 5.4 Basic difference between MATLAB and SciLab software,
- 5.5 Calculations with MATLAB or SciLab - (a) Representation of matrix (2×2 order), (b) Addition, Subtraction of matrices (2×2 order) in MATLAB or SciLab

RECOMMENDED BOOKS

1. R. D. Sharma, "Applied Mathematics – I & II for Diploma Courses", Dhanpat Rai Publications.
2. "Mathematics for Class XI", NCERT Publication, New Delhi.
3. "Mathematics for Class XII", NCERT Publication, New Delhi.
4. H. K Dass, "Applied Mathematics for Polytechnics", CBS Publishers & Distributors.
5. A Ganesh and G Balasubramanian, "Textbook of Engineering Mathematics –I", CBS Publisher, New Delhi.
6. A Ganesh and G Balasubramanian, "Textbook of Engineering Mathematics –II", CBS Publisher, New Delhi.
7. G. B. Thomas, R. L. Finney, "Calculus and Analytic Geometry", Addison Wesley, Ninth Edition.
8. B S Grewal, "Elementary Engineering Mathematics", Khanna Publishers, Delhi, Thirty-fifth Edition.
9. R.K. Jain and S.R.K. Iyengar, "Advanced Engineering Mathematics" Narosa Publishing House, New Delhi, Second Edition, 2003.
10. SS Sabharwal & Dr Sunita Jain, "Applied Mathematics Vol. I & II", Eagle Parkashan, Jalandhar.
11. S Kohli, "Engineering Mathematics Vol. I & II", IPH, Jalandhar.
12. Reena Garg & Chandrika Prasad, "Advanced Engineering Mathematics", Khanna Publishing House, New Delhi.
13. R. Pratap, "Getting Started with MATLAB 7", Oxford University Press, Seventh Edition.
14. E-books/e-tools/relevant software to be used as recommended by AICTE/HSBTE/NITTTR.

SUGGESTED WEBSITES

1. <https://www.scilab.org>

2. <http://swayam.gov.in>

INSTRUCTIONAL STRATEGY

This is theoretical subject and contains five units of equal weight age. Basic elements of Differential Calculus, Integral Calculus, and Differential Equations can be taught in the light of their applications in the field of engineering and technology. By laying more stress on applied part, teachers can also help in providing continuing education base to the students. Students need to be taught the skills needed to use software tools built by experts through multiple problem solving based on the topics that the industry requires. For example they need to know how to use mathematical models that use integration as opposed to learning how integration can be used. Useful authenticated software MATLAB or open source software SciLab can be taught theoretically by books/online literatures and basic operations can be shown practically with practical software laboratory or small mobile apps of these software or authentic Trial version of MATLAB/ SciLab software. Diploma students need to know which tools to use and how to do the job.

UNIT- I

DIFFERENTIAL CALCULUS

1.1 FUNCTIONS

Definition of Function: Let A and B be two non empty sets. A rule $f: A \rightarrow B$ (read as f from A to B) is said to be a function if to each element x of A there exists a unique element y of B such that $f(x) = y$. ' y ' is called the image of ' x ' under the map f . Here x is independent variable and y is dependent variable.

We may categorize the functions according to their nature as:

Functions Types	Algebraic	Trigonometric	Inverse Trigonometric	Exponential	Logarithmic
Examples	$y = x^2 + x + 1$, $y = x^3 - 3x + 2$, $y = x + 1$ etc.	$y = \sin x$, $y = \cos x$, $y = \sec x$ etc.	$y = \tan^{-1} x$, $y = \cos^{-1} x$, $y = \cot^{-1} x$ etc.	$y = e^x$, $y = 2^x$, $y = 5^x$ etc.	$y = \log_e x$, $y = \log_5 x$, $y = \log_{10} x$ etc.

Even Function: A function $f(x)$ is said to be an even function if $f(-x) = f(x)$ for all x .
For example: $x^2, x^4 + 1, \cos(x)$ etc.

Odd Function: A function $f(x)$ is said to be an odd function if $f(-x) = -f(x)$ for all x .
For example: $x^3, \sin(x), \tan(x)$ etc.

Periodic Function: A function $f(x)$ is said to be a periodic function if it retains same value after a certain period.

For Example: $\sin(x), \tan(x)$ etc.

As $\sin(x) = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) \dots$

Therefore $\sin(x)$ is a periodic function with period 2π .

Examples to solve problems of functions:

Example 1. If $f(x) = x^2 + 1$, find $f(2)$.

Sol. Given that $f(x) = x^2 + 1$
Put $x = 2$ in function, we get
 $f(2) = 2^2 + 1 = 4 + 1 = 5$.

Example 2. If $f(x) = 2x^2 - 4x + 6$, find $\frac{f(-2)}{f(1)}$.

Sol. Given that $f(x) = 2x^2 - 4x + 6$
Put $x = -2$ in function, we get
 $f(-2) = 2(-2)^2 - 4(-2) + 6 = 2(4) + 8 + 6 = 8 + 14 = 22$
Again put $x = 1$ in function, we get
 $f(1) = 2(1)^2 - 4(1) + 6 = 2 - 4 + 6 = 4$
Therefore $\frac{f(-2)}{f(1)} = \frac{22}{4} = 5.5$

Note:

- (i) The symbol " ∞ " is called infinity.
- (ii) $\frac{a}{0}$ is not finite (where $a \neq 0$) and it is represented by ∞ .
- (iii) $\frac{a}{\infty} = 0$ if $(a \neq \infty)$.

Indeterminate Forms: The following forms are called indeterminate forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, \infty^{\infty}, 0^0, \infty^0, \infty - \infty, 0 \times \infty \text{ etc.}$$

(These forms are meaningless)

Concept of Limits:

A function $f(x)$ is said to have limit l when x tends to a , if for every positive ε (however small) there exists a positive number δ such that $|f(x) - l| < \varepsilon$ for all values of x for which $0 < |x - a| < \delta$ and it is represented as

$$\lim_{x \rightarrow a} f(x) = l$$

Some basic properties on Limits:

- (i) $\lim_{x \rightarrow a} K = K$ where K is some constant.
- (ii) $\lim_{x \rightarrow a} K \cdot f(x) = K \cdot \lim_{x \rightarrow a} f(x)$ where K is some constant.
- (iii) $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- (iv) $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- (v) $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- (vi) $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ provided that $\lim_{x \rightarrow a} g(x) \neq 0$
- (vii) $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$

Methods of finding the limits of the functions:

- 1) Direct Substitution Method
- 2) Factorization Method

Some Standard Limits Formulas:

- 1) $\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = n a^{n-1}$
- 2) $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e$
- 3) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e$
- 4) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a$
- 5) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \log_e e = 1$
- 6) $\lim_{x \rightarrow 0} \sin x = 0$
- 7) $\lim_{x \rightarrow 0} \tan x = 0$
- 8) $\lim_{x \rightarrow 0} \cos x = 1$
- 9) $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
- 10) $\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$

Some Solved Examples on limits:**Example 3.** Evaluate $\lim_{x \rightarrow -1} (1 + x + x^2 + x^3)$.**Sol.** $\lim_{x \rightarrow -1} (1 + x + x^2 + x^3) = 1 + (-1) + (-1)^2 + (-1)^3 = 1 - 1 + 1 - 1 = 0$ **Example 4.** Evaluate $\lim_{x \rightarrow -1} \frac{x^3 + 6}{x + 1}$.**Sol.** $\lim_{x \rightarrow -1} \frac{x^3 + 6}{x + 1} = \frac{(-1)^3 + 6}{-1 + 1} = \frac{-1 + 6}{0} = \frac{5}{0} = \infty$ **Example 5.** Evaluate $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$.**Sol.** $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} = \frac{2^3 - 8}{2 - 2} = \frac{8 - 8}{2 - 2} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\begin{aligned} \therefore \lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2} &= \lim_{x \rightarrow 2} \frac{x^3 - 2^3}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x^2 + 2^2 + 2x)}{x - 2} \\ &= \lim_{x \rightarrow 2} (x^2 + 2^2 + 2x) \\ &= 2^2 + 2^2 + 2(2) \\ &= 4 + 4 + 4 = 12 \end{aligned}$$

By factorization method

Example 6. Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x - \cos x)$.**Sol.** $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x - \cos x) = \sin \frac{\pi}{2} - \cos \frac{\pi}{2} = 1 - 0 = 1$ **Example 7.** Evaluate $\lim_{x \rightarrow 0} \frac{\sin 5x}{6x}$.**Sol.** $\lim_{x \rightarrow 0} \frac{\sin 5x}{6x} = \frac{\sin(5 \times 0)}{6 \times 0} = \frac{\sin(0)}{0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\therefore \lim_{x \rightarrow 0} \frac{\sin 5x}{6x} = \lim_{x \rightarrow 0} \frac{\sin 5x}{5x} \times \frac{5}{6} = 1 \times \frac{5}{6} = \frac{5}{6}$$

 $\left[\text{by } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$ **Example 8.** Evaluate $\lim_{x \rightarrow 0} \frac{9x}{\tan 3x}$.**Sol.** $\lim_{x \rightarrow 0} \frac{9x}{\tan 3x} = \frac{9 \times 0}{\tan(3 \times 0)} = \frac{0}{\tan(0)} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\therefore \lim_{x \rightarrow 0} \frac{9x}{\tan 3x} = \lim_{x \rightarrow 0} \frac{9x}{\frac{\tan 3x}{3x} \times 3x} = \lim_{x \rightarrow 0} \frac{9x}{1 \times 3x} = \frac{9}{3} = 3$$

 $\left[\text{by } \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]$ **Examples based on trigonometric formulas:**

$$\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2} \quad \& \quad \sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}$$

Example 9. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x}$.

Sol. $\lim_{x \rightarrow 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} = \frac{\sin 0 - \sin 0}{\sin 0 + \sin 0} = \frac{0 - 0}{0 + 0} = \frac{0}{0}$ $\left(\frac{0}{0} \text{ form}\right)$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{\sin 4x - \sin 2x}{\sin 7x + \sin 3x} &= \lim_{x \rightarrow 0} \frac{2 \cos \left(\frac{4x+2x}{2}\right) \sin \left(\frac{4x-2x}{2}\right)}{2 \sin \left(\frac{7x+3x}{2}\right) \cos \left(\frac{7x-3x}{2}\right)} \\ &= \lim_{x \rightarrow 0} \frac{\cos 3x \sin x}{\sin 5x \cos 2x} = \lim_{x \rightarrow 0} \frac{\cos 3x \times \frac{\sin x}{x} \times x}{\frac{\sin 5x}{5x} \times 5x \times \cos 2x} \\ &= \lim_{x \rightarrow 0} \frac{\cos 0 \times 1 \times x}{1 \times 5x \times \cos 0} = \lim_{x \rightarrow 0} \frac{1 \times 1}{1 \times 5 \times 1} = \frac{1}{5} \end{aligned}$$

Examples based on $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a :$

Example 10. Evaluate $\lim_{x \rightarrow 0} \frac{2^x - 1}{x}$.

Sol. Applying the above formula, we get

$$\lim_{x \rightarrow 0} \frac{2^x - 1}{x} = \log_e 2 \quad (\text{here } a=2)$$

Example 11. Evaluate $\lim_{x \rightarrow 0} \frac{4^x - 3^x}{\sin x}$.

Sol. $\therefore \lim_{x \rightarrow 0} \frac{4^x - 3^x}{\sin x} = \lim_{x \rightarrow 0} \frac{4^x - 1 + 1 - 3^x}{\frac{\sin x}{x} \times x} = \lim_{x \rightarrow 0} \frac{(4^x - 1) - (3^x - 1)}{1 \times x}$

$$= \lim_{x \rightarrow 0} \left[\left(\frac{4^x - 1}{x} \right) - \left(\frac{3^x - 1}{x} \right) \right]$$

$$= \log_e 4 - \log_e 3 = \log_e \frac{4}{3}$$

$$\left(\because \log_e a - \log_e b = \log_e \frac{a}{b} \right)$$

Example 12. Evaluate $\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x \tan x}$.

Sol. $\lim_{x \rightarrow 0} \frac{e^{x^2} - 1}{x \tan x} = \lim_{x \rightarrow 0} \left[\frac{e^{x^2} - 1}{x^2} \times \frac{x^2}{x \tan x} \right]$

$$= \log_e e \times \lim_{x \rightarrow 0} \frac{x^2}{x \tan x} = 1 \times \lim_{x \rightarrow 0} \frac{x}{\tan x} = 1 \quad (\because \log_e e = 1)$$

EXERCISE-I

- If $f(x) = x^3 + 2x^2 - 3x + 1$, find $f(-1)$.
- If $f(x) = x^2 + x + 1$, find $f(2)$. $f(3)$.
- The limit of $f(x) = \frac{\cos x}{x}$ as $x \rightarrow 0$ is
a. 1 b. 0 c. -1 d. Does not exist
- Evaluate $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x + 2}$.

5. Evaluate $\lim_{x \rightarrow 5} \frac{x^2 - 25}{x - 5}$.
6. Evaluate $\lim_{x \rightarrow 3} \frac{x - 3}{x^2 - 9}$.
7. Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - 4x + 3}$.
8. Evaluate $\lim_{x \rightarrow 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3}$.
9. Evaluate $\lim_{x \rightarrow 4} \frac{x^3 - 64}{x - 4}$.
10. Evaluate $\lim_{x \rightarrow 2} \frac{\sqrt{x} - \sqrt{2}}{x - 2}$.
11. Evaluate $\lim_{x \rightarrow 5} \frac{x^{\frac{1}{3}} - 5^{\frac{1}{3}}}{x - 5}$.
12. Evaluate $\lim_{x \rightarrow 0} \frac{4x}{\sin 2x}$.
13. Evaluate $\lim_{x \rightarrow 0} \frac{\tan 6x}{3x}$.
14. Evaluate $\lim_{x \rightarrow 0} \frac{\tan 4x}{\sin 2x}$.
15. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x - \tan 2x}{x}$.
16. Evaluate $\lim_{x \rightarrow 0} \frac{\tan 4x + 2x}{2x}$.
17. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 6x + \sin 4x}{\sin 5x - \sin 3x}$.
18. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 4x - x}{6x - \sin 3x}$.
19. Evaluate $\lim_{x \rightarrow 0} \frac{5^x - 1}{x}$.
20. Evaluate $\lim_{x \rightarrow 0} \frac{3^x - 2^x}{x}$.
21. Evaluate $\lim_{x \rightarrow 0} \frac{5^x - 3^x}{\tan x}$.
22. Evaluate $\lim_{x \rightarrow 0} \frac{a^x - 1}{b^x - 1}$.
23. Evaluate $\lim_{x \rightarrow 0} \frac{2^x - 1}{3^x - 1}$.
24. Evaluate $\lim_{x \rightarrow 0} \frac{a^{\tan x} - 1}{\tan x}$.
25. Evaluate $\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$.

ANSWERS

- | | | | |
|------------------------|-----------------------------|------------------------------------|--------------------------|
| 1. 5 | 2. 91 | 3. d | 4. 0 |
| 5. 10 | 6. $\frac{1}{6}$ | 7. $-\frac{3}{2}$ | 8. $\frac{1}{2\sqrt{3}}$ |
| 9. 48 | 10. $\frac{1}{2\sqrt{2}}$ | 11. $\frac{1}{3(5)^{\frac{2}{3}}}$ | 12. 2 |
| 13. 2 | 14. 2 | 15. 1 | 16. 3 |
| 17. 5 | 18. 1 | 19. $\log_e 5$ | 20. $\log \frac{3}{2}$ |
| 21. $\log \frac{5}{3}$ | 22. $\frac{\log a}{\log b}$ | 23. $\frac{\log 2}{\log 3}$ | 24. $\log_e a$ |
| 25. 1 | | | |

1.2 DIFFERENTIATION

Increment: Increment is the quantity by which the value of a variable changes. It may be positive or negative. e.g. suppose the value of a variable x changes from 5 to 5.3 then 0.3 is the increment in x . Similarly, if the value of variable x changes from 5 to 4.5 then -0.5 is the increment in x .

Usually δx represents the increment in x , δy represents the increment in y , δz represents the increment in z etc.

Derivative or Differential Co-efficient: If y is a function of x . Let δx be the increment in x and δy be the corresponding increment in y , then $\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$ (if it exists) is called the derivative or differential co-efficient of y with respect to x and is denoted by $\frac{dy}{dx}$.

i.e.
$$\frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x}$$

Differentiation:

Let $y = f(x)$ (1)

Let δx be the increment in x and δy be the corresponding increment in y , then

$$y + \delta y = f(x + \delta x) \quad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = f(x + \delta x) - f(x)$$

$$\Rightarrow \delta y = f(x + \delta x) - f(x)$$

Dividing both sides by δx , we get

$$\frac{\delta y}{\delta x} = \frac{f(x + \delta x) - f(x)}{\delta x}$$

Taking limit $\delta x \rightarrow 0$ on both sides, we get

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

If this limit exists, we write it as

$$\frac{dy}{dx} = f'(x)$$

$$\text{where } f'(x) = \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x}.$$

This is called the differentiation or derivative of the function $f(x)$ with respect to x .

The above method is also known as **First Principle Method** of differentiation.

Example 1. Differentiate the following by first principle method of differentiation:

$$(i) \quad x^n \qquad (ii) \quad \sin x$$

$$\text{Sol. (i) Let } y = x^n \qquad (1)$$

Let δx be the increment in x and δy be the corresponding increment in y , then

$$y + \delta y = (x + \delta x)^n \qquad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = (x + \delta x)^n - x^n$$

$$\Rightarrow \delta y = (x + \delta x)^n - x^n$$

Dividing both sides by δx , we get

$$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^n - x^n}{\delta x}$$

Taking limit $\delta x \rightarrow 0$ on both sides, we get

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{(x + \delta x)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{x^n \left(1 + \frac{\delta x}{x} \right)^n - x^n}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{x^n \left[\left(1 + \frac{\delta x}{x} \right)^n - 1 \right]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{x^n \left[1 + n \left(\frac{\delta x}{x} \right) + \frac{n(n-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots - 1 \right]}{\delta x}$$

(used Binomial Expansion)

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{x^n \left[n \left(\frac{\delta x}{x} \right) + \frac{n(n-1)}{2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right]}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} x^n \left[\frac{n}{\delta x} \left(\frac{\delta x}{x} \right) + \frac{n(n-1)}{\delta x \cdot 2!} \left(\frac{\delta x}{x} \right)^2 + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} x^n \left[\frac{n}{x} + \frac{n(n-1)}{2!} \frac{\delta x}{x^2} + \dots \right]$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} x^n \left[\frac{n}{x} \right]$$

$$\Rightarrow \frac{dy}{dx} = n \cdot x^{n-1}$$

$$(ii) \quad \text{Let } y = \sin x \quad (1)$$

Let δx be the increment in x and δy be the corresponding increment in y , then

$$y + \delta y = \sin(x + \delta x) \quad (2)$$

Subtracting equation (1) from equation (2), we get

$$y + \delta y - y = \sin(x + \delta x) - \sin x$$

$$\Rightarrow \delta y = \sin(x + \delta x) - \sin x$$

Dividing both sides by δx , we get

$$\frac{\delta y}{\delta x} = \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

Taking limit $\delta x \rightarrow 0$ on both sides, we get

$$\lim_{\delta x \rightarrow 0} \frac{\delta y}{\delta x} = \lim_{\delta x \rightarrow 0} \frac{\sin(x + \delta x) - \sin x}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{2 \cos\left(\frac{x + \delta x + x}{2}\right) \sin\left(\frac{x + \delta x - x}{2}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{2 \cos\left(\frac{2x + \delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \frac{2 \cos\left(\frac{2x + \delta x}{2}\right) \sin\left(\frac{\delta x}{2}\right)}{2 \cdot \left(\frac{\delta x}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \lim_{\delta x \rightarrow 0} \cos\left(\frac{2x + \delta x}{2}\right) \cdot \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)}$$

$$\Rightarrow \frac{dy}{dx} = \cos\left(\frac{2x + 0}{2}\right) \cdot 1$$

$$\Rightarrow \frac{dy}{dx} = \cos x$$

$$\text{i.e. } \frac{d}{dx}(\sin x) = \cos x$$

$$\left[\because \lim_{\delta x \rightarrow 0} \frac{\sin\left(\frac{\delta x}{2}\right)}{\left(\frac{\delta x}{2}\right)} = 1 \right]$$

Notations: The first order derivative of the function $f(x)$ with respect to x can be represented in the following ways:

$$\frac{d}{dx}(f(x)), \frac{df}{dx}, f'(x), f_1(x) \text{ etc.}$$

Similarly, the first order derivative of y with respect to x can be represented as:

$$\frac{dy}{dx}, y', y_1 \text{ etc.}$$

Physical Interpretation of Derivatives:

Let the variable t represents the time and the function $f(t)$ represents the distance travelled in time t .

We know that $Speed = \frac{\text{Distance travelled}}{\text{Time taken}}$

If time interval is between ' a ' & ' $a+h$ '. Here h be increment in a . Then the speed in that interval is given by

$$\begin{aligned} & \frac{\text{Distance travelled upto time } (a+h) - \text{Distance travelled upto time } (a)}{\text{length of time interval}} \\ &= \frac{f(a+h) - f(a)}{a+h-a} = \frac{f(a+h) - f(a)}{h} \end{aligned}$$

If we take $h \rightarrow 0$ then $\frac{f(a+h) - f(a)}{h}$ approaches the speed at time $t=a$. Thus we can say that derivative is related in the similar way as speed is related to the distance travelled by a moving particle.

Some Properties of Differentiation:

If $f(x)$ and $g(x)$ are differentiable functions, then

- (i) $\frac{d}{dx}(K) = 0$ where K is some constant.
- (ii) $\frac{d}{dx}(K \cdot f(x)) = K \cdot \frac{d}{dx}(f(x))$ where K is some constant.
- (iii) $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$
- (iv) $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$
- (v) $\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$

This property is known as Product Rule of differentiation.

$$(vi) \quad \frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \frac{d}{dx}(g(x))}{(g(x))^2} \quad \text{provided that } g(x) \neq 0$$

This property is known as Quotient Rule of differentiation.

Differentiation of some basic functions:

- (i) $\frac{d}{dx}(x^n) = n x^{n-1}$. This is known as power formula, here n is any real number.
- (ii) $\frac{d}{dx}(a^x) = a^x \log_e a$ here $a > 0$ & $a \neq 1$
- (iii) $\frac{d}{dx}(e^x) = e^x \log_e e = e^x$

$$(iv) \quad \frac{d}{dx}(\log_e x) = \frac{1}{x}$$

$$(v) \quad \frac{d}{dx}(\log_a x) = \frac{1}{x} \log_a e$$

Example 2. Differentiate $y = x^{10}$ with respect to x .

Sol. Given that $y = x^{10}$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^{10}) = 10x^9$$

Example 3. Differentiate $y = x^{-2}$ with respect to x .

Sol. Given that $y = x^{-2}$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^{-2}) = -2 \cdot x^{-2-1} = -2 \cdot x^{-3} = -\frac{2}{x^3}$$

1.3 DIFFERENTIATION OF SUM, PRODUCT AND QUOTIENT OF FUNCTIONS

Differentiation of sum of two or more functions, product of two or more functions and quotient of two or more functions are explained with following examples as:

Example 4. Differentiate $y = 5 - x^6$ with respect to x .

Sol. Given that $y = 5 - x^6$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(5 - x^6) = \frac{d}{dx}(5) - \frac{d}{dx}(x^6) \\ &= 0 - 6x^5 = -6x^5 \end{aligned}$$

Example 5. Differentiate $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ with respect to x .

Sol. Given that $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}\left(\frac{1}{\sqrt{x}}\right) \\ &= \frac{d}{dx}\left(x^{\frac{1}{2}}\right) - \frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} - \left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1} \\ &= \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{3}{2}} \end{aligned}$$

Example 6. Differentiate $y = e^x \cdot a^x + 2x^3 - \log x$ with respect to x .

Sol. Given that $y = e^x \cdot a^x + 2x^3 - \log x = (ea)^x + 2x^3 - \log x$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}((ea)^x + 2x^3 - \log x) = \frac{d}{dx}(ea)^x + 2 \frac{d}{dx}(x^3) - \frac{d}{dx}(\log x)$$

$$= (ea)^x \log_e(ea) + 2(3x^2) - \frac{1}{x} = (ea)^x \log_e(ea) + 6x^2 - \frac{1}{x}$$

Example 7. Differentiate $y = x^2 \cdot e^x$ with respect to x .

Sol. Given that $y = x^2 \cdot e^x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^2 \cdot e^x) \\ \Rightarrow \frac{dy}{dx} &= x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2) \\ \Rightarrow \frac{dy}{dx} &= x^2 \cdot e^x + e^x \cdot 2x \\ \Rightarrow \frac{dy}{dx} &= e^x(x^2 + 2x) \end{aligned}$$

Example 8. Differentiate $y = x^{-3} \cdot \log x$ with respect to x .

Sol. Given that $y = x^{-3} \cdot \log x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x^{-3} \cdot \log x) \\ \Rightarrow \frac{dy}{dx} &= x^{-3} \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^{-3}) \\ \Rightarrow \frac{dy}{dx} &= x^{-3} \cdot \frac{1}{x} + \log x \cdot (-3x^{-4}) \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{x^4} - 3 \frac{\log x}{x^4} \\ \Rightarrow \frac{dy}{dx} &= \frac{1}{x^4}(1 - 3 \log x) \end{aligned}$$

Example 9. Differentiate $y = \frac{\log x}{x}$ with respect to x .

Sol. Given that $y = \frac{\log x}{x}$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\log x}{x}\right) \\ \Rightarrow \frac{dy}{dx} &= \frac{x \frac{d}{dx}(\log x) - \log x \frac{d}{dx}(x)}{x^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{1 - \log x}{x^2} \end{aligned}$$

Example 10. Differentiate $y = \frac{x^2}{e^x}$ with respect to x .

Sol. Given that $y = \frac{x^2}{e^x}$

Differentiating it with respect to x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{x^2}{e^x} \right) \\ \Rightarrow \frac{dy}{dx} &= \frac{e^x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(e^x)}{e^{2x}} \\ \Rightarrow \frac{dy}{dx} &= \frac{e^x \cdot 2x - x^2 \cdot e^x}{e^{2x}} \\ \Rightarrow \frac{dy}{dx} &= \frac{e^x \cdot (2x - x^2)}{e^{2x}} \\ \Rightarrow \frac{dy}{dx} &= \frac{2x - x^2}{e^x}\end{aligned}$$

Chain Rule: If $f(x)$ and $g(x)$ are two differentiable functions then

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot \frac{d}{dx}(g(x)) = f'(g(x)) \cdot g'(x)$$

EXERCISE-II

- Differentiate the following by first principle method of differentiation:
(i) e^x (ii) $\tan x$
- Differentiate $y = \sqrt{x}$ with respect to x .
- Differentiate $y = x^{-\frac{5}{2}}$ with respect to x .
- Differentiate $y = 2 - x + 3x^2$ with respect to x .
- Differentiate $y = (x+3)(x-1)$ with respect to x .
- Differentiate $y = \frac{x^2 + 7}{x}$ with respect to x .

ANSWERS

- | | | |
|--------------------------|-----------------------------------|--------------|
| 1. (i) e^x | (ii) $\sec^2 x$ | |
| 2. $\frac{1}{2\sqrt{x}}$ | 3. $-\frac{5}{2}x^{-\frac{7}{2}}$ | 4. $-1 + 6x$ |
| 5. $2x + 2$ | 6. $1 - 7x^{-2}$ | |

UNIT- II

DIFFERENTIAL CALCULUS AND IT's APPLICATIONS

2.1 Differentiation of trigonometric functions, inverse trigonometric functions. Logarithmic differentiation, successive differentiation (upto 2nd order)

Differentiation of Trigonometric and Inverse Trigonometric Functions:

- (i) $\frac{d}{dx}(\sin x) = \cos x$
- (ii) $\frac{d}{dx}(\cos x) = -\sin x$
- (iii) $\frac{d}{dx}(\tan x) = \sec^2 x$
- (iv) $\frac{d}{dx}(\sec x) = \sec x \tan x$
- (v) $\frac{d}{dx}(\cos ec x) = -\cos ec x \cot x$
- (vi) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- (vii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- (viii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- (ix) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- (x) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
- (xi) $\frac{d}{dx}(\cos ec^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$

Example 1. Differentiate $y = 4x - 2 \cot^{-1} x$ with respect to x .

Sol. Given that $y = 4x - 2 \cot^{-1} x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(4x - 2 \cot^{-1} x) \\ \Rightarrow \frac{dy}{dx} &= 4 \times 1 - 2 \left(\frac{-1}{1+x^2} \right) \\ \Rightarrow \frac{dy}{dx} &= 4 + \frac{2}{1+x^2} \\ \Rightarrow \frac{dy}{dx} &= \frac{4(1+x^2) + 2}{1+x^2} \end{aligned}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4+4x^2+2}{1+x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{6+4x^2}{1+x^2}$$

Example 2. Differentiate $y = x \cos x$ with respect to x .

Sol. Given that $y = x \cos x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x \cos x) = x \cdot \frac{d}{dx}(\cos x) + \cos x \cdot \frac{dx}{dx} \\ &= x \cdot (-\sin x) + \cos x \cdot 1 = -x \sin x + \cos x \end{aligned}$$

Example 3. Differentiate $y = \log x \tan x$ with respect to x .

Sol. Given that $y = \log x \tan x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(\log x \tan x) = \log x \cdot \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(\log x) \\ &= \log x \times (\sec^2 x) + \tan x \times \frac{1}{x} = \log x \cdot \sec^2 x + \frac{\tan x}{x} \end{aligned}$$

Example 4. Differentiate $y = \frac{\sin x}{x}$ with respect to x .

Sol. Given that $y = \frac{\sin x}{x}$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot \frac{dx}{dx}}{x^2} \\ &= \frac{x \cdot \cos x - \sin x \cdot 1}{x^2} = \frac{x \cos x - \sin x}{x^2} \end{aligned}$$

Example 5. Differentiate $y = x \cdot \sin^{-1} x$ with respect to x .

Sol. Given that $y = x \cdot \sin^{-1} x$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(x \cdot \sin^{-1} x) \\ \Rightarrow \frac{dy}{dx} &= x \cdot \frac{d}{dx}(\sin^{-1} x) + \sin^{-1} x \cdot \frac{d}{dx}(x) \\ \Rightarrow \frac{dy}{dx} &= x \cdot \frac{1}{\sqrt{1-x^2}} + \sin^{-1} x \cdot 1 \\ \Rightarrow \frac{dy}{dx} &= \frac{x}{\sqrt{1-x^2}} + \sin^{-1} x \end{aligned}$$

Logarithmic Differentiation :

Let $f(x)$ and $g(x)$ are two differentiable function and $y = f(x)^{g(x)}$

To differentiate y , first we take logarithm of y :

$$\log y = \log(f(x)^{g(x)})$$

$$\log y = g(x) \log(f(x)) \quad (\log a^b = b \log a)$$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{d}{dx}(\log y) &= \frac{d}{dx}(g(x) \log(f(x))) \\ \frac{1}{y} \frac{dy}{dx} &= g(x) \frac{d}{dx}(\log f(x)) + \log f(x) \frac{d}{dx}(g(x)) \\ \frac{1}{y} \frac{dy}{dx} &= g(x) \frac{1}{f(x)} f'(x) + \log f(x) g'(x) \\ \frac{dy}{dx} &= y \left[\frac{g(x)}{f(x)} f'(x) + \log f(x) g'(x) \right] \\ \frac{dy}{dx} &= f(x)^{g(x)} \left[\frac{g(x)}{f(x)} f'(x) + \log f(x) g'(x) \right] \end{aligned}$$

Examples based on Derivative of $f(x)^{g(x)}$ Logarithmic Differentiation :

Example 6. Differentiate $y = x^x$ with respect to x .

Sol. Given that $y = x^x$

Taking logarithm on both sides, we get

$$\begin{aligned} \log y &= \log x^x \\ \log y &= x \log x \quad (\log a^b = b \log a) \end{aligned}$$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{d}{dx}(\log y) &= \frac{d}{dx}(x \log x) \\ \frac{1}{y} \frac{dy}{dx} &= x \frac{d}{dx}(\log x) + \log x \frac{dx}{dx} \\ \frac{1}{y} \frac{dy}{dx} &= x \cdot \frac{1}{x} + \log x \cdot 1 \\ \frac{1}{y} \frac{dy}{dx} &= 1 + \log x \\ \frac{dy}{dx} &= y(1 + \log x) \\ \frac{dy}{dx} &= x^x (1 + \log x) \end{aligned}$$

Example 7. Differentiate $y = x^{\sin x}$ with respect to x .

Sol. Given that $y = x^{\sin x}$

Taking logarithm on both sides, we get

$$\begin{aligned} \log y &= \log x^{\sin x} \\ \log y &= \sin x \cdot \log x \quad (\log a^b = b \log a) \end{aligned}$$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{d}{dx}(\log y) &= \frac{d}{dx}(\sin x \cdot \log x) \\ \frac{1}{y} \frac{dy}{dx} &= \sin x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(\sin x) \end{aligned}$$

$$\frac{1}{y} \frac{dy}{dx} = \sin x \cdot \frac{1}{x} + \log x \cdot \cos x$$

$$\frac{dy}{dx} = y \left(\frac{\sin x}{x} + \log x \cos x \right)$$

$$\frac{dy}{dx} = x^{\sin x} \left(\frac{\sin x}{x} + \log x \cos x \right)$$

Example 8. Differentiate $y = (\sin x)^{\cos x}$ with respect to x .

Sol. Given that $y = (\sin x)^{\cos x}$

Taking logarithm on both sides, we get

$$\log y = \log (\sin x)^{\cos x}$$

$$\log y = \cos x \log (\sin x) \quad (\log a^b = b \log a)$$

Differentiating it with respect to x , we get

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(\cos x \log (\sin x))$$

$$\frac{1}{y} \frac{dy}{dx} = \cos x \frac{d}{dx}(\log (\sin x)) + \log (\sin x) \frac{d}{dx}(\cos x)$$

$$\frac{1}{y} \frac{dy}{dx} = \cos x \frac{1}{\sin x} \frac{d}{dx}(\sin x) + \log (\sin x) \cdot (-\sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = \cot x \cos x - \sin x \log (\sin x)$$

$$\frac{dy}{dx} = y (\cot x \cos x - \sin x \log (\sin x))$$

$$\frac{dy}{dx} = (\sin x)^{\cos x} (\cot x \cos x - \sin x \log (\sin x))$$

SUCCESSIVE DIFFERENTIATION OR HIGHER ORDER DERIVATIVE

Let $y = f(x)$ be a differentiable function, then $\frac{dy}{dx}$ represents the first order derivative of y

with respect to x . If we may further differentiate it i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, then it is called second

order derivative of y with respect to x . Some other way to represent second order derivative of y with respect to x : $\frac{d^2 y}{dx^2}$, y'' , y_2 .

So, successive derivatives of y with respect to x can be represented as

$$\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \dots, \frac{d^n y}{dx^n} \text{ etc.}$$

Example 9. If $y = x^8 - 12x^5 + 5x^3 - 12$, find $\frac{d^2 y}{dx^2}$.

Sol. Given that $y = x^8 - 12x^5 + 5x^3 - 12$

Differentiating with respect to x , we get

$$\frac{d y}{d x} = \frac{d}{d x} (x^8 - 12 x^5 + 5 x^3 - 12)$$

$$\Rightarrow \frac{d y}{d x} = 8 x^7 - 60 x^4 + 15 x^2$$

Again differentiating with respect to x , we get

$$\frac{d^2 y}{d x^2} = \frac{d}{d x} (8 x^7 - 60 x^4 + 15 x^2)$$

$$\Rightarrow \frac{d^2 y}{d x^2} = 56 x^6 - 240 x^3 + 30 x$$

Example 10. If $f(x) = x^2 \cdot \sin x$, find $f'(0)$ and $f''\left(\frac{\pi}{2}\right)$.

Sol. Given that $f(x) = x^2 \cdot \sin x$

Differentiating with respect to x , we get

$$\frac{d}{d x} (f(x)) = \frac{d}{d x} (x^2 \cdot \sin x)$$

$$\Rightarrow f'(x) = x^2 \frac{d}{d x} (\sin x) + \sin x \frac{d}{d x} (x^2)$$

$$\Rightarrow f'(x) = x^2 \cos x + \sin x \cdot 2x$$

$$\Rightarrow f'(x) = x^2 \cos x + 2x \sin x \quad (1)$$

Again differentiating with respect to x , we get

$$f''(x) = \frac{d}{d x} (x^2 \cos x + 2x \sin x)$$

$$\Rightarrow f''(x) = \frac{d}{d x} (x^2 \cos x) + 2 \frac{d}{d x} (x \sin x)$$

$$\Rightarrow f''(x) = x^2 \frac{d}{d x} (\cos x) + \cos x \frac{d}{d x} (x^2) + 2 \left[x \frac{d}{d x} (\sin x) + \sin x \frac{d}{d x} (x) \right]$$

$$\Rightarrow f''(x) = -x^2 \sin x + 2x \cos x + 2[x \cos x + \sin x]$$

$$\Rightarrow f''(x) = -x^2 \sin x + 4x \cos x + 2 \sin x \quad (2)$$

Put $x = 0$ in equation (1), we get

$$f'(0) = (0)^2 \cos 0 + 2 \times 0 \times \sin 0 = 0$$

Put $x = \frac{\pi}{2}$ in equation (2), we get

$$f''\left(\frac{\pi}{2}\right) = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 4 \times \frac{\pi}{2} \times \cos\left(\frac{\pi}{2}\right) + 2 \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow f''\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{4} + 2$$

Example 11. If $y = \sin Ax + \cos Ax$, prove that $\frac{d^2 y}{d x^2} + A^2 y = 0$.

Soln. Given that $y = \sin Ax + \cos Ax$

Differentiating with respect to x , we get

$$\frac{d y}{d x} = \frac{d}{d x} (\sin Ax + \cos Ax)$$

$$\Rightarrow \frac{d y}{d x} = A \cos Ax - A \sin Ax$$

Again differentiating with respect to x , we get

$$\begin{aligned}\frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}(A \cos Ax - A \sin Ax) \\ \Rightarrow \frac{d^2y}{dx^2} &= -A^2 \sin Ax - A^2 \cos Ax \\ \Rightarrow \frac{d^2y}{dx^2} &= -A^2 (\sin Ax + \cos Ax) \\ \Rightarrow \frac{d^2y}{dx^2} &= -A^2 y \\ \Rightarrow \frac{d^2y}{dx^2} + A^2 y &= 0\end{aligned}$$

Example 12. If $y = e^{-Ax}$, prove that $\frac{d^2y}{dx^2} + A \frac{dy}{dx} = 0$.

Soln. Given that $y = e^{-Ax}$

Differentiating with respect to x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(e^{-Ax}) \\ \Rightarrow \frac{dy}{dx} &= e^{-Ax} \frac{d}{dx}(-Ax) \\ \Rightarrow \frac{dy}{dx} &= -A e^{-Ax} \\ \Rightarrow \frac{dy}{dx} &= -A y\end{aligned}$$

Again differentiating with respect to x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= -A \frac{dy}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} + A \frac{dy}{dx} &= 0\end{aligned}$$

EXERCISE-I

1. Differentiate $y = x^2 \sin x$ with respect to x .
2. Differentiate $y = \cos x \log x$ with respect to x .
3. Differentiate $y = (3t^2 - 9)2^t$ with respect to t .
4. Differentiate $y = \frac{\log x}{\tan x}$ with respect to x .
5. Differentiate $y = \frac{x^2 + 1}{\sin x}$ with respect to x .
6. Differentiate $y = (\cos x)^x$ with respect to x .
7. Differentiate $y = x^{\cos x}$ with respect to x .
27. If $y = \log(\sin x) + e^{5x}$, find $\frac{d^2y}{dx^2}$.

28. If $y = x^3 \cdot e^{-2x}$, find $\frac{d^2 y}{d x^2}$ at $x = 3$.
29. If $y = \sin x + x$, find $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$
- a. $\frac{\pi}{2}$ b. $-\frac{\pi}{2}$ c. 1 d. 0

ANSWERS

1. $x^2 \cos x + 2x \sin x$ 2. $\frac{\cos x}{x} - \log x \cdot \sin x$
3. $2^t \{ (3t^2 - 9) \log_e 2 + 6t \}$ 4. $\frac{\tan x - x \log x \cdot \sec^2 x}{x \tan^2 x}$
5. $\frac{2x \sin x - (x^2 + 1) \cos x}{\sin^2 x}$ 6. $(\cos x)^x (-x \tan x + \log(\cos x))$
7. $x^{\cos x} \left(\frac{\cos x}{x} - \log x \sin x \right)$
27. $-\operatorname{cosec}^2 x + 25e^{5x}$ 28. $18e^{-6}$ 29. c

2.2 Application of differential calculus in: (a) Rate measures (b) Maxima and minima
(a) Derivative as Rate Measures:

Let y be a function of x , then $\frac{dy}{dx}$ represents the rate of change of y with respect to x .

If $\frac{dy}{dx} > 0$ then y increases when x changes and if $\frac{dy}{dx} < 0$ then y decreases when x changes.

Some Important Points to Remember:

- (i) Usually t , s , v and a are used to represent time, displacement, velocity and acceleration respectively.

Also
$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = v \cdot \frac{dv}{ds}$$

- (ii) If the particle moves in the direction of s increasing, then $v = \frac{ds}{dt} > 0$ and if the particle moves in the direction of s decreasing, then $v = \frac{ds}{dt} < 0$.
- (iii) If $a = 0$ then the particle is said to be moving with constant velocity and if $a < 0$ then the particle is said to have retardation.
- (iv) If $\frac{dy}{dx} = 0$ then y is constant.
- (v) If $y = f(x)$ be a curve then $\frac{dy}{dx}$ is said to be the slope of the curve. It is also represented by m i.e. $\text{slope} = m = \frac{dy}{dx}$.

- (vi) If r is the radius, A is the area and C is the circumference of the circle then $A = \pi r^2$ & $C = 2\pi r$.
- (vii) If r is the radius, S is the surface area and V is the volume of the sphere then $S = 4\pi r^2$ & $V = \frac{4}{3}\pi r^3$.
- (viii) If r is the radius of base, h is the height, l is the slant length, S is the surface area and V is the volume of the cone then $S = \pi r l + \pi r^2$ & $V = \frac{1}{3}\pi r^2 h$.
- (ix) If a is the length of side of a base, S is the surface area and V is the volume of the cube then $S = 6a^2$ & $V = a^3$.

Examples Related to Rate Measure:

Example 1. If $y = x^3 + 5x^2 - 6x + 7$ and x increases at the rate of 3 units per minute, how fast is the slope of the curve changes when $x = 2$ units.

Sol. Let t represents the time.

$$\text{Given that } y = x^3 + 5x^2 - 6x + 7 \quad (1)$$

$$\text{and } \frac{dx}{dt} = 3 \quad (2)$$

Let m be the slope of the curve.

$$\therefore m = \frac{dy}{dx}$$

$$\Rightarrow m = \frac{d}{dx}(x^3 + 5x^2 - 6x + 7) \quad (\text{using (1)})$$

$$\Rightarrow m = 3x^2 + 10x - 6$$

Differentiating it with respect to t , we get

$$\begin{aligned} \frac{dm}{dt} &= \frac{d}{dt}(3x^2 + 10x - 6) \\ \Rightarrow \frac{dm}{dt} &= (6x + 10) \frac{dx}{dt} \\ \Rightarrow \frac{dm}{dt} &= (6x + 10) \cdot 3 \quad (\text{using (2)}) \end{aligned}$$

$$\Rightarrow \frac{dm}{dt} = 18x + 30 \quad (3)$$

Put $x = 2$ in (3), we get

$$\left(\frac{dm}{dt} \right)_{x=2} = 18(2) + 30 = 36 + 30 = 66$$

Hence the slope of given curve increases at the rate of 66 units/minute when $x = 2$.

Example 2. A particle is moving along a straight line such that the displacement s after time t is given by $s = 2t^2 + t + 7$. Find the velocity and acceleration at time $t = 20$.

Sol. Let v be the velocity and a be the acceleration of the particle at time t .

Given that the displacement of the particle is $s = 2t^2 + t + 7$

Differentiating it with respect to t , we get

$$\frac{ds}{dt} = \frac{d}{dt}(2t^2 + t + 7)$$

$$\Rightarrow v = 4t + 1 \quad (1)$$

Again differentiating with respect to t , we get

$$\frac{dv}{dt} = \frac{d}{dt}(4t + 1)$$

$$\Rightarrow a = 4 \quad (2)$$

Put $t = 20$ in (1) and (2), we get

$$[v]_{t=20} = 4(20) + 1 = 81 \quad \& \quad [a]_{t=20} = 4$$

Hence velocity of the particle is 81 and acceleration is 4 when $t = 20$.

Example 3. Find the rate of change of the area of the circle with respect to its radius r when $r = 4$ cm.

Sol. Given that r be the radius of the circle.

Let A be the area of the circle.

$$\therefore A = \pi r^2$$

$$\Rightarrow \frac{dA}{dr} = \frac{d}{dr}(\pi r^2)$$

$$\Rightarrow \frac{dA}{dr} = 2\pi r \quad (1)$$

Put $r = 4$ in (1), we get

$$\left(\frac{dA}{dr}\right)_{r=4} = 2\pi \times 4 = 8\pi$$

Hence the rate of change of area of the circle is 8π c.m..

Example 4. Find the rate of change of the surface area of a ball with respect to its radius r .

Sol. Given that r is the radius of the ball.

Let S be the surface area of the ball.

$$\therefore S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dr} = \frac{d}{dr}(4\pi r^2)$$

$$\Rightarrow \frac{dS}{dr} = 4\pi \times 2r = 8\pi r$$

which is the required rate of change of the surface area of a ball with respect to its radius r .

Example 5. The radius of an air bubble increases at the rate of 2 cm/sec. At what rate is the volume of the bubble increases when the radius is 5 cm?

Sol. Let r be the radius, V be the volume of the bubble and t represents time.

$$\text{So, by given statement } \frac{dr}{dt} = 2 \text{ cm/sec} \quad (1)$$

$$\text{and } V = \frac{4}{3}\pi r^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right)$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \times \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 8\pi r^2 \text{ c.m./sec} \quad (2)$$

Put $r = 5 \text{ c.m.}$ in (2), we get

$$\left(\frac{dV}{dt} \right)_{r=5} = 8\pi \times (5)^2 = 200\pi \text{ c.m.}^3/\text{sec}$$

Hence, volume of the bubble increases at the rate of $200\pi \text{ cm}^3/\text{sec}$.

Example 6. Find the rate of change of the volume of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone.

$$\therefore V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \frac{dV}{dr} = \frac{d}{dr} \left(\frac{1}{3}\pi r^2 h \right)$$

$$\Rightarrow \frac{dV}{dr} = \frac{1}{3}\pi h \times 2r = \frac{2}{3}\pi r h.$$

Example 7. Find the rate of change of the surface area of the cone with respect to the radius of its base.

Sol. Let r be the radius of the base, l be the slant length and S be the surface area of the cone.

$$\therefore S = \pi r l + \pi r^2$$

$$\Rightarrow \frac{dS}{dr} = \frac{d}{dr} (\pi r l + \pi r^2)$$

$$\Rightarrow \frac{dS}{dr} = \pi l + \pi \times 2r$$

$$\Rightarrow \frac{dS}{dr} = \pi l + 2\pi r$$

Example 8. Sand is pouring from a pipe at the rate 10 cc/sec . The falling sand forms a cone on the ground in such a way that the height of the cone is always one-fifth of the radius of the base. How fast the height of the sand cone increases when the height is 6 cm ?

Sol. Let r be the radius of the base, h be the height and V be the volume of the cone at the time t .

$$\text{So, by given statement } h = \frac{r}{5} \quad (1)$$

$$\text{and } V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow V = \frac{1}{3}\pi (5h)^2 h = \frac{25}{3}\pi h^3$$

(using (1))

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt} \left(\frac{25}{3}\pi h^3 \right) = \frac{25}{3}\pi \times 3h^2 \frac{dh}{dt} = 25\pi h^2 \frac{dh}{dt} \quad (2)$$

$$\text{Also, by given statement } \frac{dV}{dt} = 10 \text{ cc/sec} \quad (3)$$

From (2) and (3), we get

$$25\pi h^2 \frac{dh}{dt} = 10$$

$$\Rightarrow \frac{dh}{dt} = \frac{10}{25\pi h^2} = \frac{2}{5\pi h^2}$$

$$\text{When } h=6\text{ cm}, \quad \frac{dh}{dt} = \frac{2}{5\pi(6)^2} = \frac{1}{90\pi}$$

Hence, the rate of increase of height of the sand cone is $\frac{1}{90\pi} \text{ cm/sec}$, when $h=6\text{ cm}$.

Example 9. The length of edges of a cube increases at the rate of 2 cm/sec . At what rate is the volume of the cube increases when the edge length is 1 cm ?

Sol. Let a be the length of edge and V be the volume of the cube at time t .

$$\text{So, by given statement } \frac{da}{dt} = 2\text{ cm/sec} \quad (1)$$

$$\text{and } V = a^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{d}{dt}(a^3)$$

$$\Rightarrow \frac{dV}{dt} = 3a^2 \frac{da}{dt} \quad (\text{using (1)})$$

$$\Rightarrow \frac{dV}{dt} = 6a^2 \quad (2)$$

Put $a=1$ in (2), we get

$$\left(\frac{dV}{dt}\right)_{a=1} = 6(1)^2 = 6$$

Hence, volume of the cube increases at the rate of $6\text{ cm}^3/\text{sec}$.

(b) Maxima and Minima

Maximum Value of a Function & Point of Maxima: Let $f(x)$ be a function defined on domain $D \subset R$. Let a be any point of domain D . We say that $f(x)$ has maximum value at a if $f(x) \leq f(a)$ for all $x \in D$ and a is called the point of maxima.

e.g. Let $f(x) = -x^2 + 5$ for all $x \in R$

Now $x^2 \geq 0$ for all $x \in R$

$$-x^2 \leq 0 \quad \text{for all } x \in R$$

$$-x^2 + 5 \leq 5 \quad \text{for all } x \in R$$

i.e. $f(x) \leq 5$ for all $x \in R$

Hence 5 is the maximum value of $f(x)$ which is attained at $x=0$. Therefore, $x=0$ is the point of maxima.

Minimum Value of a Function & Point of Minima: Let $f(x)$ be a function defined on domain $D \subset R$. Let a be any point of domain D . We say that $f(x)$ has minimum value at a if $f(x) \geq f(a)$ for all $x \in D$ and a is called the point of minima.

e.g. Let $f(x) = x^2 + 8$ for all $x \in R$

Now $x^2 \geq 0$ for all $x \in R$

$$x^2 + 8 \geq 8 \quad \text{for all } x \in R$$

i.e. $f(x) \geq 8$ for all $x \in R$

Hence 8 is the minimum value of $f(x)$ which is attained at $x=0$. Therefore, $x=0$ is the point of minima.

Note: We can also attain points of maxima and minima and their corresponding maximum and minimum value of a given function by differential calculus too.

Working Rule to find points of maxima or minima or inflexion by Differential Calculus:

Step No.	Working Procedure
1	Put $y = f(x)$
2	Find $\frac{dy}{dx}$
3	Put $\frac{dy}{dx} = 0$ and solve it for x . Let $x_1, x_2, \dots, x_n$ are the values of x .
4	Find $\frac{d^2y}{dx^2}$.
5	Put the values of x in $\frac{d^2y}{dx^2}$. Suppose $x=x_i$ be any value of x . If $\frac{d^2y}{dx^2} < 0$ at $x=x_i$ then $x=x_i$ is the point of maxima and $f(x_i)$ is the maximum value of $f(x)$. If $\frac{d^2y}{dx^2} > 0$ at $x=x_i$ then $x=x_i$ is the point of minima and $f(x_i)$ is minimum value of $f(x)$. If $\frac{d^2y}{dx^2} = 0$ at $x=x_i$. Find $\frac{d^3y}{dx^3}$. If $\frac{d^3y}{dx^3} \neq 0$ at $x=x_i$ then $x=x_i$ is the point of inflexion.

Examples of Maxima and Minima:

Example 10. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 - 12x^2 + 5$.

Sol. Let $y = f(x) = x^3 - 12x^2 + 5$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 12x^2 + 5)$$

$$\frac{dy}{dx} = 3x^2 - 24x$$

Again differentiating with respect to x , we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(3x^2 - 24x)$$

$$\frac{d^2y}{dx^2} = 6x - 24$$

Put $\frac{dy}{dx} = 0$, we get

$$3x^2 - 24x = 0$$

$$3x(x-8) = 0$$

$$\text{Either } x = 0 \text{ or } x - 8 = 0$$

$$\text{i.e. } \text{Either } x = 0 \text{ or } x = 8$$

When $x = 0$:

$$\left(\frac{d^2 y}{dx^2} \right)_{x=0} = (6x - 24)_{x=0} = -24 < 0$$

which shows that $x = 0$ is a point of maxima.

So maximum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=0} = (x^3 - 12x^2 + 5)_{x=0} = 5$$

When $x = 8$:

$$\left(\frac{d^2 y}{dx^2} \right)_{x=8} = (6x - 24)_{x=8} = 6(8) - 24 = 24 > 0$$

which shows that $x = 8$ is a point of minima.

So minimum value of $f(x) = x^3 - 12x^2 + 5$ is

$$(y)_{x=8} = (x^3 - 12x^2 + 5)_{x=8} = 8^3 - 12(8)^2 + 5 \\ = 512 - 768 + 5 = -251$$

Example 11. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = \sin x + \cos x$ where $0 \leq x \leq \frac{\pi}{2}$.

Sol. Let $y = f(x) = \sin x + \cos x$

Differentiating it with respect to x , we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x + \cos x)$$

$$\frac{dy}{dx} = \cos x - \sin x$$

Again differentiating with respect to x , we get

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}(\cos x - \sin x)$$

$$\frac{d^2 y}{dx^2} = -\sin x - \cos x$$

Put $\frac{dy}{dx} = 0$, we get

$$\cos x - \sin x = 0$$

$$\cos x = \sin x$$

$$\frac{\cos x}{\sin x} = 1$$

$$\cot x = 1$$

$$\cot x = \cot \frac{\pi}{4}$$

$$\text{as } 0 \leq x \leq \frac{\pi}{2}$$

$$x = \frac{\pi}{4}$$

When $x = \frac{\pi}{4}$:

$$\begin{aligned} \left(\frac{d^2 y}{d x^2} \right)_{x=\frac{\pi}{4}} &= (-\sin x - \cos x)_{x=\frac{\pi}{4}} = -\sin \frac{\pi}{4} - \cos \frac{\pi}{4} \\ &= -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} < 0 \end{aligned}$$

which shows that $x = \frac{\pi}{4}$ is a point of maxima.

So maximum value of $f(x) = \sin x + \cos x$ is

$$\begin{aligned} (y)_{x=\frac{\pi}{4}} &= (\sin x + \cos x)_{x=\frac{\pi}{4}} \\ &= \sin \frac{\pi}{4} + \cos \frac{\pi}{4} \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} \end{aligned}$$

Example 12. Find two positive numbers x & y such that $x \cdot y = 16$ and the sum $x + y$ is minimum. Also find the minimum value of sum.

Sol. Given that $x \cdot y = 16 \Rightarrow y = \frac{16}{x}$

Let $S = x + y \Rightarrow S = x + \frac{16}{x}$

Differentiating it with respect to x , we get

$$\begin{aligned} \frac{dS}{dx} &= \frac{d}{dx} \left(x + \frac{16}{x} \right) \\ \frac{dS}{dx} &= 1 + 16 \left(-\frac{1}{x^2} \right) = 1 - \frac{16}{x^2} \end{aligned}$$

Again differentiating with respect to x , we get

$$\begin{aligned} \frac{d^2 S}{dx^2} &= \frac{d}{dx} \left(1 - \frac{16}{x^2} \right) \\ \frac{d^2 S}{dx^2} &= 0 - 16 \left(-\frac{2}{x^3} \right) = \frac{32}{x^3} \end{aligned}$$

Put $\frac{dS}{dx} = 0$, we get

$$\begin{aligned} 1 - \frac{16}{x^2} &= 0 \\ \frac{x^2 - 16}{x^2} &= 0 \\ x^2 - 16 &= 0 \end{aligned}$$

Either $x = 4$ or $x = -4$

$x = -4$ is rejected as x is positive.

When $x = 4$:

$$\left(\frac{d^2 S}{dx^2} \right)_{x=4} = \left(\frac{32}{x^3} \right)_{x=4} = \frac{32}{4^3} = \frac{32}{64} = \frac{1}{2} > 0$$

which shows that $x = 4$ is a point of minima.

Now at $x=4$, value of y is :

$$(y)_{x=4} = \left(\frac{16}{x}\right)_{x=4} = \frac{16}{4} = 4$$

Also minimum value of sum $S = x + y$ is

$$(S)_{x=4, y=4} = (x+y)_{x=4, y=4} = 4+4=8$$

Example 13. Find the dimensions of the rectangle of given area 169 sq. cm. whose perimeter is least. Also find its perimeter.

Sol. Let x c.m. and y c.m. are the sides, A sq.c.m. is the area and P c.m. be the perimeter of the rectangle.

$$\therefore A = xy = 169 \quad \Rightarrow \quad y = \frac{169}{x}$$

$$\text{And } P = 2(x + y) \quad \Rightarrow \quad P = 2\left(x + \frac{169}{x}\right) = 2x + \frac{338}{x}$$

Differentiating it with respect to x , we get

$$\frac{dP}{dx} = \frac{d}{dx}\left(2x + \frac{338}{x}\right)$$

$$\frac{dP}{dx} = 2 + 338\left(-\frac{1}{x^2}\right) = 2 - \frac{338}{x^2}$$

Again, differentiating with respect to x , we get

$$\frac{d^2P}{dx^2} = \frac{d}{dx}\left(2 - \frac{338}{x^2}\right)$$

$$\frac{d^2P}{dx^2} = 0 - 338\left(-\frac{2}{x^3}\right) = \frac{676}{x^3}$$

Put $\frac{dP}{dx} = 0$, we get

$$2 - \frac{338}{x^2} = 0$$

$$\frac{2x^2 - 338}{x^2} = 0$$

$$2x^2 - 338 = 0$$

$$x^2 = 169$$

Either $x = 13$ or $x = -13$

$x = -13$ is rejected as x can't be negative.

When $x = 13$:

$$\left(\frac{d^2P}{dx^2}\right)_{x=13} = \left(\frac{676}{x^3}\right)_{x=13} = \frac{676}{(13)^3} = \frac{4}{13} > 0$$

which shows that $x = 13$ is a point of minima.

Therefore, Perimeter is least at $x = 13$.

Now at $x = 13$, value of y is :

$$(y)_{x=13} = \left(\frac{169}{x}\right)_{x=13} = \frac{169}{13} = 13$$

Also least value of perimeter is

$$(P)_{x=13, y=13} = (2x + 2y)_{x=13, y=13} = 26 + 26 = 52 \text{ cm}$$

Example 14. Show that among all the rectangles of a given perimeter, the square has the maximum area.

Sol. Let the sides of the rectangle are x and y , A be the area and P be the given perimeter.

$$\therefore P = 2(x + y) \Rightarrow P = 2x + 2y \Rightarrow y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

and

$$A = xy = \frac{Px}{2} - x^2$$

Differentiating it with respect to x , we get

$$\frac{dA}{dx} = \frac{d}{dx} \left(\frac{Px}{2} - x^2 \right)$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x$$

Again differentiating with respect to x , we get

$$\frac{d^2A}{dx^2} = \frac{d}{dx} \left(\frac{P}{2} - 2x \right)$$

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

Put $\frac{dA}{dx} = 0$, we get

$$\frac{P}{2} - 2x = 0$$

$$x = \frac{P}{4}$$

When $x = \frac{P}{4}$:

$$\left(\frac{d^2A}{dx^2} \right)_{x=\frac{P}{4}} = -2 < 0$$

which shows that $x = \frac{P}{4}$ is a point of maxima.

Therefore, Area is maximum at $x = \frac{P}{4}$.

Now at $x = \frac{P}{4}$, value of y is:

$$(y)_{x=\frac{P}{4}} = \left(\frac{P}{2} - x \right)_{x=\frac{P}{4}} = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$$\Rightarrow x = y = \frac{P}{4} \text{ gives the maximum area.}$$

Hence, among all the rectangles of a given perimeter, the square has the maximum area.

Example 15. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = x^3 + 1$.

Sol. Let $y = f(x) = x^3 + 1$

Differentiating it with respect to x , we get

$$\frac{d y}{d x} = \frac{d}{d x}(x^3 + 1)$$

$$\frac{d y}{d x} = 3x^2$$

Again, differentiating with respect to x , we get

$$\frac{d^2 y}{d x^2} = \frac{d}{d x}(3x^2)$$

$$\frac{d^2 y}{d x^2} = 6x$$

Put $\frac{d y}{d x} = 0$, we get

$$3x^2 = 0$$

$$\Rightarrow x = 0$$

When $x = 0$:

$$\left(\frac{d^2 y}{d x^2} \right)_{x=0} = (6x)_{x=0} = 6 \times 0 = 0$$

To check maxima or minima, we need to find third order derivative of y with respect to x .

$$\text{So, } \frac{d^3 y}{d x^3} = \frac{d}{d x}(6x)$$

$$\Rightarrow \frac{d^3 y}{d x^3} = 6 \neq 0$$

which shows that $x = 0$ is neither a point of maxima nor a point of minima, hence the given function has neither maximum value nor minimum value.

EXERCISE-II

1. If $y = 5 - 3x^2 + 2x^3$ and x decreases at the rate of 6 units per second, how fast is the slope of the curve changes when $x = 7$.
2. If a particle is moving in a straight line such that the displacement s after time t is given by $s = \frac{1}{2}vt$, where v be the velocity of the particle. Prove that the acceleration a of the particle is constant.
3. The radius of the circle increases at the rate 0.4 cm/sec . What is the increase of its circumference.
4. Find the rate of change per second of the volume of a ball with respect to its radius r when $r = 6 \text{ cm}$.
5. Find the rate of change per minute of the surface area of a ball with respect to its radius r when $r = 9 \text{ m}$.
6. Find the rate of change of the volume of the cone with respect to its height.
7. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = 6x^3 - 27x^2 + 36x + 6$.
8. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = -2x^3 + 6x^2 + 18x - 1$.

9. Find all the points of maxima and minima and the corresponding maximum and minimum values of the function $f(x) = \frac{\log x}{x}$ if $0 < x < \infty$.
10. The maximum value of the function $y = -x^3 + 1$ over the interval $[-2, 1]$
 a. 9 b. 0 c. 2 d. 10
11. Find the length (l) and breadth (b) of a rectangle with perimeter 12 such that it has maximum area.
 a. 3,3 b. 2,4 c. 3,4 d. 5,2

ANSWERS

1. -468 units/second 3. 0.8π cm/sec 4. 144π cm³/sec
5. 72π m²/min 6. $\frac{1}{3}\pi r^2$, where r is the radius of the base of the cone
7. $x=1$ is a point of maxima and maximum value of function is 21, $x=2$ is a point of minima and minimum value of function is 18.
8. $x=-1$ is a point of minima and minimum value of function is -11, $x=3$ is a point of maxima and maximum value of function is 53.
9. $x=e$ is a point of maxima and maximum value of function is $\frac{1}{e}$.
10. a 11. a

* * * * *

UNIT- III INTEGRAL CALCULUS

3.1 Integration as inverse operation of differentiation

Integration is the reverse process of differentiation. In the previous chapter, we have studied differentiation as the study of small change in one variable with respect to small change in other. In the same manner, integration is the study of a function as a whole when small changes are given. For example

if dA shows small change in area, then $\int dA$ is the area as a whole.

If $f(x)$ and $g(x)$ are two functions such that $\frac{dg(x)}{dx} = f(x)$, then $\int f(x)dx = g(x) + c$ i.e. $g(x)$ is integral of $f(x)$ with respect to x and c is constant of integration.

The function to be integrated is called integrand and put in between the sign $\int dx$.

Main Rule of Integration: We can integrate a function if it is in single form (variables/functions are not in product or quotient form) otherwise we will have to use various different methods to convert it in single form and then integrate.

Some Results of Integration

$$I. \quad \frac{d}{dx} \int f(x)dx = f(x) + c$$

$$II. \quad \int kf(x)dx = k \int f(x)dx \quad \text{where } k \text{ is any constant.}$$

$$III. \quad \int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$$

$$IV. \quad \int kdx = kx + c \quad \text{where } k \text{ is any constant and } c \text{ constant of integration}$$

3.2 Simple standard integrals and related problems, Integration by Substitution method and Integration by parts

(a) Integral of algebraic functions:

$$\int 0 \, dx = \text{constant}$$

$$\int 1 \, dx = x + c$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \, n \neq -1, \, n \text{ is any real number}$$

$$\int \frac{1}{x} \, dx = \log |x| + c$$

Example 1. Evaluate (i) $\int x^3 \, dx$ (ii) $\int \sqrt{x} \, dx$ (iii) $\int \frac{1}{\sqrt{x}} \, dx$ (iv) $\int 2 \, dx$

(v) $\int (2x^2 + 3x + 5) \, dx$ (vi) $\int (x^3 - 5x + \frac{1}{x}) \, dx$

Sol : (i) $\int x^3 \, dx = \frac{x^4}{4} + c$

(iv) $\int 2 \, dx = 2x + c$

(ii) $\int \sqrt{x} \, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$

(v) $\int (2x^2 + 3x + 5) \, dx = 2 \frac{x^3}{3} + 3 \frac{x^2}{2} + 5x + c$

(iii) $\int \frac{1}{\sqrt{x}} \, dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

(vi) $\int (x^3 - 5x + \frac{1}{x}) \, dx = \frac{x^4}{4} - 5 \frac{x^2}{2} + \log x + c$

Example 2. Evaluate (i) $\int (x+1)x^2 \, dx$ (ii) $\int \frac{x^2+1}{x} \, dx$ (iii) $\int \frac{x^3-x+3}{x} \, dx$

(iv) $\int (\sqrt{x} + \frac{1}{\sqrt{x}}) \, dx$

Sol : (i) $\int (x+1)x^2 \, dx = \int (x^3 + x^2) \, dx = \frac{x^4}{4} + \frac{x^3}{3} + c$

(ii) $\int \frac{x^2+1}{x} \, dx = \int \left(x + \frac{1}{x} \right) \, dx = \frac{x^2}{2} + \log x + c$

(iii) $\int \frac{x^3-x+3}{x} \, dx = \int \left(x^2 - 1 + \frac{3}{x} \right) \, dx = \frac{x^3}{3} - x + \log x + c$

$$(iv) \int (\sqrt{x} + \frac{1}{\sqrt{x}}) dx = \frac{x^{3/2}}{3/2} + \frac{x^{1/2}}{1/2} + c$$

EXERCISE-I

Integrate the following functions with respect to x

1. Evaluate

- i. $\int x^4 dx$ ii. $\int x^{\frac{5}{4}} dx$ iii. $\int \frac{1}{x^5} dx$ iv. $\int \frac{1}{x^{3/4}} dx$
2. $\int (3x\sqrt{x} + 4\sqrt{x} + 5) dx$
3. $\int (\sqrt{x} + \frac{1}{\sqrt{x}})^2 dx$
4. $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx$
5. $\int \left(x + \frac{1}{x}\right)^3 dx$
6. $\int \frac{(1+x)^2}{\sqrt{x}} dx$
7. $\int \frac{(x+1)(x-2)}{\sqrt{x}} dx$

ANSWERS

1. (i) $\frac{x^5}{5} + c$ (ii) $\frac{4}{9}x^{\frac{9}{4}} + c$ (iii) $\frac{x^{-4}}{-4} + c$ (iv) $4x^{\frac{1}{4}} + c$
2. $\frac{6}{5}x^{\frac{5}{2}} + \frac{8}{3}x^{\frac{3}{2}} + 5x + c$ 3. $\frac{x^2}{2} + \log|x| + 2x + c$

$$4. \frac{x^2}{2} + 5x + 4 \log |x| - \frac{1}{x} + c$$

$$5. \frac{x^4}{4} - \frac{1}{2x^2} + 3\frac{x^2}{2} + 3 \log |x| + c$$

$$6. 2x^{\frac{1}{2}} + \frac{2}{5}x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$$

$$7. \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + c$$

(b) Integrals of the type $(ax + b)^n$

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{(n+1)a} + c, n \neq -1, n \text{ is any real number}$$

$$\int \frac{1}{ax + b} dx = \frac{\log |ax + b|}{a} + c$$

Example 3. Evaluate (i) $\int (1 - 3x)^5 dx$

(ii) $\int \sqrt{2x + 3} dx$

(iii) $\int \frac{1}{3 + 2x} dx$ (iv) $\int \frac{1}{(5 + 3x)^3} dx$

Sol : (i) $\int (1 - 3x)^5 dx = \frac{(1 - 3x)^6}{6(-3)} + c = \frac{(1 - 3x)^6}{-18} + c$

(ii) $\int \sqrt{2x + 3} dx = \frac{(2x + 3)^{3/2}}{\frac{3}{2}} + c = \frac{(2x + 3)^{3/2}}{3} + c$

(iii) $\int \frac{1}{3 + 2x} dx = \frac{1}{2} \log |(3 + 2x)| + c$

(iv) $\int \frac{1}{(5 + 3x)^3} dx = \int (5 + 3x)^{-3} dx = \frac{(5 + 3x)^{-2}}{-2 \times 3} + c = \frac{(5 + 3x)^{-2}}{-6} + c$

EXERCISE-II

Integrate the following functions with respect to x

1. $\int \sqrt{3 - 2x} dx$

2. $\int (5+3x)^{-7} dx$

3. $\int (5x+3)^4 dx$

4. $\int \frac{1}{\sqrt{1-4x}} dx$

5. $\int \frac{x^2+5x+2}{x+2} dx$

6. $\int \left(\frac{1}{2-3x} + \frac{1}{\sqrt{3x-2}} dx \right)$

7. $\int \sqrt{3+2x} + (3x-5)^4 dx$

8. $\int \frac{x-1}{\sqrt{x+4}} dx$

ANSWERS

1. $\frac{(3-2x)^{3/2}}{-3} + c$

2. $\frac{(5+3x)^{-8}}{-24} + c$

3. $\frac{(5x+3)^5}{25} + c$

4. $\frac{(1-4x)^{1/2}}{-2} + c$

5. $\frac{x^2}{2} + 3x - 4 \log |x+2| + c$

6. $\frac{\log |2-3x|}{-3} + \frac{2(3x-2)^{1/2}}{3} + c$

7. $\frac{(3+2x)^{3/2}}{3} + \frac{(3x-5)^5}{15} + c$

8. $\frac{2(x+4)^{3/2}}{3} - 10(x+4)^{1/2} + c$

(c) More formulae of integrals of the following type:

1. $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

2. $\int \frac{-1}{x^2+a^2} dx = \frac{1}{a} \cot^{-1} \frac{x}{a} + c$

$$3. \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log |x + \sqrt{x^2 + a^2}| + c$$

$$4. \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| + c$$

$$5. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

$$6. \int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1} \frac{x}{a} + c$$

$$7. \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c$$

$$8. \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a + x}{a - x} \right| + c$$

$$9. \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \frac{x}{a} + c$$

$$10. \int \frac{-1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{cosec}^{-1} \frac{x}{a} + c$$

Example 4. Evaluate (i) $\int \frac{dx}{x^2 + 9}$ (ii) $\int \frac{dx}{2x^2 + 9}$ (iii) $\int \frac{dx}{\sqrt{1 - 3x^2}}$

(iv) $\int \frac{dx}{\sqrt{4x^2 + 16}}$ (v) $\int \frac{dx}{x^2 - 25}$ (vi) $\int \frac{dx}{9x^2 - 16}$

Sol : (i) $\int \frac{dx}{x^2 + 9} = \int \frac{dx}{x^2 + 3^2}$

Using formula $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

Here $a = 3$

So $\int \frac{dx}{x^2 + 3^2} = \frac{1}{3} \tan^{-1} \frac{x}{3} + c$

$$(ii) \quad \int \frac{dx}{2x^2+9} = \frac{1}{2} \int \frac{dx}{\frac{2x^2+9}{2}} = \frac{1}{2} \int \frac{dx}{x^2 + \frac{9}{2}} = \frac{1}{2} \int \frac{dx}{x^2 + \left(\frac{3}{\sqrt{2}}\right)^2}$$

Using formula $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$

$$a = \frac{3}{\sqrt{2}}$$

$$\begin{aligned} \therefore \int \frac{dx}{x^2 + \left(\frac{3}{\sqrt{2}}\right)^2} &= \frac{1}{2} \left[\frac{1}{3/\sqrt{2}} \tan^{-1} \frac{x}{3/\sqrt{2}} \right] + c \\ &= \frac{1}{3\sqrt{2}} \tan^{-1} \frac{\sqrt{2}x}{3} + c \end{aligned}$$

$$\begin{aligned} (iii) \quad \int \frac{dx}{\sqrt{1-9x^2}} &= \int \frac{dx}{\sqrt{\left(\frac{1-9x^2}{9}\right)9}} = \int \frac{dx}{3\sqrt{\frac{1}{9}-x^2}} \\ &= \frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} \quad \text{Here } a = \frac{1}{3} \end{aligned}$$

Using formula $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a} + c$

$$\frac{1}{3} \int \frac{dx}{\sqrt{\left(\frac{1}{3}\right)^2 - x^2}} = \frac{1}{3} \left[\sin^{-1} \frac{x}{1/3} \right] + c = \frac{1}{3} [\sin^{-1} 3x] + c$$

$$\begin{aligned} (iv) \quad \int \frac{dx}{\sqrt{4x^2+16}} &= \int \frac{dx}{\sqrt{\left(\frac{4x^2+16}{4}\right)4}} \\ &= \int \frac{dx}{2\sqrt{x^2+4}} = \frac{1}{2} \int \frac{dx}{\sqrt{x^2+2^2}} \end{aligned}$$

Here $a = 2$. Using formula $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| + c$

$$\frac{1}{2} \int \frac{dx}{\sqrt{x^2 + 2^2}} = \frac{1}{2} [\log |x + \sqrt{x^2 + 2^2}|] + c$$

$$(v) \quad \int \frac{dx}{\sqrt{x^2 - 25}} = \int \frac{dx}{\sqrt{x^2 - 5^2}}$$

Using formula $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log |x + \sqrt{x^2 - a^2}| + c$

$$\therefore \int \frac{dx}{\sqrt{x^2 - 5^2}} = \log |x + \sqrt{x^2 - 5^2}| + c$$

$$(vi) \quad \int \frac{dx}{9x^2 - 16} = \int \frac{dx}{\frac{9x^2 - 16}{9} \times 9} = \frac{1}{9} \int \frac{dx}{x^2 - \frac{16}{9}}$$

$$\frac{1}{9} \int \frac{dx}{x^2 - \left(\frac{4}{3}\right)^2} \cdot \text{Here } a = \frac{4}{3}$$

Using formula $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left| \frac{x - a}{x + a} \right| + c$

$$\therefore \frac{1}{9} \int \frac{dx}{x^2 - \left(\frac{4}{3}\right)^2} = \frac{1}{9} \left[\frac{1}{2 \times \frac{4}{3}} \log \left| \frac{x - \frac{4}{3}}{x + \frac{4}{3}} \right| \right] + c$$

$$= \frac{1}{9} \left[\frac{3}{8} \log \frac{3x - 4}{3x + 4} \right] + c$$

$$= \frac{1}{24} \log \frac{3x - 4}{3x + 4} + c$$

EXERCISE-III

1. Integrate the following integral with respect to x .

$$\begin{array}{lll} \text{(i)} \int \frac{dx}{\sqrt{9-x^2}} & \text{(ii)} \int \frac{dx}{x\sqrt{x^2-25}} & \text{(iii)} \int \frac{dx}{\sqrt{16x^2-36}} \\ \text{(iv)} \int \frac{dx}{25x^2-16} & \text{(v)} \int \frac{dx}{x^2+49} & \text{(vi)} \int \frac{dx}{4+9x^2} \end{array}$$

2. Evaluate $\int \frac{x^2}{1+x^2} dx$

3. Integrate $\frac{x^6+2}{x^2+1}$ with respect to x.

4. Integrate $\left(1 + \frac{1}{1+x^2} - \frac{2}{\sqrt{1-x^2}} + \frac{5}{x\sqrt{x^2-1}}\right)$ with respect to x.

ANSWERS

$$1. \text{(i)} \sin^{-1} \frac{x}{3} + c \quad \text{(ii)} \frac{1}{5} \sec^{-1} \frac{x}{5} + c \quad \text{(iii)} \frac{1}{4} \log \left| x + \sqrt{x^2 - \left(\frac{3}{2}\right)^2} \right| + c$$

$$\text{(iv)} \frac{1}{40} \log \left| \frac{5x-4}{5x+4} \right| + c \quad \text{(v)} \frac{1}{7} \tan^{-1} \frac{x}{7} + c \quad \text{(vi)} \frac{1}{6} \tan^{-1} \frac{3x}{2} + c$$

$$2. x - \tan^{-1} x + c \quad 3. \frac{x^5}{5} - \frac{x^3}{3} + x + \tan^{-1} x + c$$

$$4. x + \tan^{-1} x - 2 \sin^{-1} x + 5 \sec^{-1} x + c$$

(d) Integrals of the Exponential functions

An exponential function is of the form (constant)^{variable}, i.e. e^x , a^x , b^{3x} etc.

$$\text{(i)} \int a^{mx} dx = \frac{a^{mx}}{(\log a)m} + c$$

$$\text{(ii)} \int e^{mx} dx = \frac{e^{mx}}{m} + c$$

Example 5. Evaluate (i) $\int 3^x dx$ (ii) $\int \frac{a^x}{b^x} dx$

$$(iii) \int \frac{a^x + b^x}{a^x b^x} dx \quad (iv) \int \frac{2^{3x} b^x}{e^x} dx$$

Sol : (i) $\int 3^x dx = \frac{3^x}{(\log 3).1} + c$

(ii) $\int \frac{a^x}{b^x} dx = \int \left(\frac{a}{b} \right)^x dx = \frac{(a/b)^x}{\log(a/b)} + c$

(iii) $\int \frac{a^x + b^x}{a^x b^x} dx = \int \left(\frac{a^x}{a^x b^x} + \frac{b^x}{a^x b^x} \right) dx$

$$= \int \left(\frac{1}{b^x} + \frac{1}{a^x} \right) dx$$

$$= \int (b^{-x} + a^{-x}) dx$$

$$= \frac{b^{-x}}{(\log b)(-1)} + \frac{a^{-x}}{(\log a)(-1)} + c$$

(iv) $\int \frac{2^{3x} b^x}{e^x} dx = \int \left(\frac{2^3 b}{e} \right)^x dx = \int \left(\frac{8b}{e} \right)^x dx$

$$= \frac{\left(\frac{8b}{e} \right)^x}{\log \left(\frac{8b}{e} \right).1} + c$$

EXERCISE -IV

1. Evaluate $\int (a^{2x} + b^{2x}) dx$.
2. Integrate $\frac{(a^x + b^x)^2}{a^x b^x}$ with respect to x.
3. Integrate $\left(\frac{1}{b^x} - \frac{1}{a^x} \right)$ with respect to x.

4. Evaluate $\int (x^a + e^x + e^a) dx$.
5. Integrate $\frac{2^x + 3^x}{5^x}$ with respect to x.

ANSWERS

1. $\frac{a^{2x}}{(\log a)2} + \frac{b^{2x}}{(\log b)2} + c$
2. $\frac{\left(\frac{a}{b}\right)^x}{\log \frac{a}{b}} + \frac{\left(\frac{b}{a}\right)^x}{\log \frac{b}{a}} + 2x + c$
3. $-\frac{b^{-x}}{(\log b)} + \frac{a^{-x}}{(\log a)} + c$
4. $\frac{x^{a+1}}{a+1} + \frac{e^x}{1} + e^a x + c$
5. $\frac{\left(\frac{2}{5}\right)^x}{\log \frac{2}{5}} + \frac{\left(\frac{3}{5}\right)^x}{\log \frac{3}{5}} + c$

(e) Integral of Trigonometric Functions

1. $\int \sin x \, dx = -\cos x + c$
2. $\int \cos x \, dx = \sin x + c$
3. $\int \tan x \, dx = \log |\sec x| + c$ or $-\log |\cos x| + c$
4. $\int \operatorname{cosec} x \, dx = \log |\operatorname{cosec} x - \cot x| + c$
5. $\int \sec x \, dx = \log |\sec x + \tan x| + c$
6. $\int \cot x \, dx = \log |\sin x| + c$ or $-\log |\operatorname{cosec} x| + c$
7. $\int \sec^2 x \, dx = \tan x + c$
8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$
9. $\int \sec x \tan x \, dx = \sec x + c$
10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

Example 6. Evaluate : (i) $\int \tan^2 x \, dx$ (ii) $\int \cot^2 x \, dx$

(iii) $\int \sin 3x \cos 2x \, dx$ (iv) $\int \frac{2+3\cos x}{\sin^2 x} dx$

(v) $\int \frac{1}{1+\sin x} dx$ (vi) $\int \frac{dx}{1-\cos x}$

Sol : (i) $\int \tan^2 x \, dx = \int (\sec^2 x - 1) dx = \int \sec^2 x \, dx - \int dx$

$$= \tan x - x + c$$

(ii) $\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) dx$

$$= \int \operatorname{cosec}^2 x \, dx - \int dx = -\cot x - x + c$$

(iii) $\int \sin 3x \cos 2x \, dx = \frac{1}{2} \int 2 \sin 3x \cos 2x \, dx$

Using formula $2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

$$= \frac{1}{2} \int [\sin(3x + 2x) + \sin(3x - 2x)] dx$$

$$= \frac{1}{2} \int (\sin 5x + \sin x) dx$$

$$= \frac{1}{2} \left[-\frac{\cos 5x}{5} - \frac{\cos x}{1} \right] + c$$

(iv) $\int \frac{2+3\cos x}{\sin^2 x} dx$

$$= \int \frac{2}{\sin^2 x} dx + \int \frac{3\cos x}{\sin^2 x} dx$$

$$= 2 \int \operatorname{cosec}^2 x \, dx + 3 \int \cot x \operatorname{cosec} x \, dx$$

$$= -2 \cot x - 3 \operatorname{cosec} x + c$$

$$\begin{aligned}
 \text{(v)} \quad & \int \frac{dx}{1+\sin x} \\
 &= \int \frac{dx}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} \\
 &= \int \frac{1-\sin x}{1-\sin^2 x} dx \quad \text{as } (a-b)(a+b) = a^2 - b^2 \\
 &= \int \frac{1-\sin x}{\cos^2 x} dx \\
 &= \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx \\
 &= \int \sec^2 x dx - \int \tan x \sec x dx \\
 &= \tan x - \sec x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(vi)} \quad & \int \frac{dx}{1-\cos x} \\
 &= \int \frac{1}{1-\cos x} \times \frac{1+\cos x}{1+\cos x} dx \\
 &= \int \frac{1+\cos x}{1-\cos^2 x} dx \\
 &= \int \frac{1+\cos x}{\sin^2 x} dx \\
 &= \int \frac{1}{\sin^2 x} dx + \int \frac{\cos x}{\sin^2 x} dx \\
 &= \int \operatorname{cosec}^2 x dx + \int \cot x \operatorname{cosec} x dx \\
 &= -\cot x - \operatorname{cosec} x + c
 \end{aligned}$$

EXERCISE-V

1. The integral of $\frac{1}{x^2+9}$ with respect to x is

(a) $\tan^{-1}\left(\frac{x}{3}\right) + c$

(b) $\frac{1}{9}\tan^{-1}\left(\frac{x}{9}\right) + c$

(c) $-\frac{1}{3}\cot^{-1}\left(\frac{x}{9}\right) + c$

(d) $\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$

2. The value of $\int \sec^2 x \, dx$ is

(a) $\frac{1}{2}\log(x^2 + 1) + c$

(b) $\log(x^2 + 1) + c$

(c) $\tan x + c$

(d) $\cot x + c$

3. Evaluate $\int (3\sin x - 4\cos x + \frac{5}{\cos^2 x} - \frac{6}{\sin^2 x} + \sec x) dx$.

4. Evaluate $\int (\sec^2 x + \operatorname{cosec}^2 x) dx$.

5. Evaluate $\int \frac{5\cos^2 x + 6\sin^2 x}{2\sin^2 x \cos^2 x} dx$.

6. Evaluate $\int (\tan x + \cot x)^2 dx$.

7. Evaluate $\int \frac{dx}{1 + \cos x}$.

8. Evaluate $\int \frac{dx}{1 - \sin x}$.

9. Evaluate $\int 2\cos 3x \cos x \, dx$.

10. Evaluate $\int \sin 5x \sin 2x \, dx$.

11. Evaluate $\int \frac{\sin x}{1 + \sin x} dx$.

12. Evaluate $\int \frac{\sec x}{\sec x + \tan x} dx$.

ANSWERS

1. (d)

2. (c)

3. $-3 \cos x - 4 \sin x + 5 \tan x + 6 \cot x + \log |\sec x + \tan x| + c$

4. $\tan x - \cot x + c$ 5. $-\frac{5}{2} \cot x + 3 \tan x + c$

6. $\tan x - \cot x + c$ 7. $-\cot x + \operatorname{cosec} x + c$

8. $\tan x + \sec x + c$ 9. $\frac{\sin 4x}{4} + \frac{\sin 2x}{2} + c$

10. $\frac{1}{2} \left[-\frac{\sin 7x}{7} + \frac{\sin 3x}{3} \right] + c$ 11. $x - \tan x + \sec x + c$

12. $\tan x - \sec x + c$

There are various methods to convert the product/quotient of functions into a single function like Multiple/divide/splitting/Use of formulae. If a given function can't be integrated by the methods explained till now:

(f) Substitution method

(g) Integration by parts

(f) Integration by Substitution method: Many problems of the integration may be solved by method of substitution. Two fundamental deductions are given below:

1. $\int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$
2. $\int (f(x))^n f'(x) dx = \frac{(f(x))^{n+1}}{n+1} + c$ where $n \neq -1$

Note: Here we substituted $f(x) = t$ and hence put $f'(x)dx = dt$. After substitution we have integrated the function.

Example 7. Evaluate $\int \frac{2x+3}{x^2+3x-8} dx$.

Sol. Put $x^2 + 3x - 8 = t$
 $\Rightarrow \frac{d}{dx}(x^2 + 3x - 8) = \frac{dt}{dx}$
 $\Rightarrow 2x + 3 - 0 = \frac{dt}{dx}$
 $\Rightarrow (2x + 3)dx = dt$

We have

$$\begin{aligned} \int \frac{2x+3}{x^2+3x-8} dx &= \int \frac{dt}{t} \\ &= \log|t| + c \\ &= \log|x^2 + 3x - 8| + c \end{aligned}$$

Example 8. Evaluate $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$.

Sol. Put $e^x - e^{-x} = t$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(e^x - e^{-x}) &= \frac{dt}{dx} \\
\Rightarrow e^x + e^{-x} &= \frac{dt}{dx} \\
\Rightarrow (e^x + e^{-x})dx &= dt \\
\text{We have} \\
\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx &= \int \frac{dt}{t} \\
&= \log|t| + c \\
&= \log|e^x - e^{-x}| + c
\end{aligned}$$

Example 9. Evaluate $\int \frac{1}{x \cdot (\log x)^2} dx$.

Sol. Put $\log x = t$

$$\begin{aligned}
\Rightarrow \frac{d}{dx}(\log x) &= \frac{dt}{dx} \\
\Rightarrow \frac{1}{x} &= \frac{dt}{dx} \\
\Rightarrow \frac{1}{x} dx &= dt \\
\text{We have} \\
\int \frac{1}{x \cdot (\log x)^2} dx &= \int \frac{dt}{t^2} \\
&= \int t^{-2} dt \\
&= \frac{t^{-1}}{-1} + c \\
&= -\frac{1}{t} + c \\
&= -\frac{1}{\log x} + c
\end{aligned}$$

(g) Integration by Parts Method

If $f(x)$ and $g(x)$ are two functions of x then

$$\int f(x)g(x)dx = f(x) \int g(x)dx - \int \left\{ \frac{d}{dx} f(x) \int g(x)dx \right\} dx.$$

Here in above formula $f(x)$ is chosen as first function and $g(x)$ is chosen as second function.

Note : Selection of I function and II function in accordance with function which comes first in the word “ILATE”.

where I = Inverse functions

L = Logarithmic functions

A = Algebraic functions

T = Trigonometric functions

E = Exponential functions

Example 10. Evaluate (i) $\int x \sin x \, dx$ (ii) $\int x \tan^2 x \, dx$ (iii) $\int \log x \, dx$

Sol : (i) $\int x \sin x \, dx = x \int \sin x \, dx - \int \left\{ \frac{d}{dx} x \int \sin x \, dx \right\} dx$

$$= x(-\cos x) - \int 1\{-\cos x\} dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + c$$

(ii) $\int x \tan^2 x \, dx = \int x(\sec^2 x - 1) dx$

$$= \int x \sec^2 x \, dx - \int x \, dx$$

$$= x \int \sec^2 x \, dx - \int \left\{ \frac{d}{dx} x \int \sec^2 x \, dx \right\} dx - \frac{x^2}{2}$$

$$= x \tan x - \int \{1 \tan x\} dx - \frac{x^2}{2}$$

$$= x \tan x - \int \tan x \, dx - \frac{x^2}{2}$$

$$= x \tan x - \log \sec x - \frac{x^2}{2} + c$$

(iii) $\int \log x \, dx = \int (1 \log x) dx$

$$= \int x^0 \log x \, dx$$

$$= \log x \int x^0 dx - \int \left\{ \frac{d}{dx} \log x \int x^0 dx \right\} dx$$

$$= (\log x)x - \int \left(\frac{1}{x} x \right) dx$$

$$= (\log x)x - \int 1 \, dx$$

$$= (\log x)x - x + c$$

Example 11. Evaluate (i) $\int x^2 \cos x \, dx$ (ii) $\int x^2 e^x \, dx$.

Sol : (i) $\int x^2 \cos x \, dx = x^2 \int \cos x \, dx - \int \left\{ \frac{d}{dx} x^2 \int \cos x \, dx \right\} dx$

$$= x^2 \sin x - \int 2\{x \sin x\} dx$$

$$= x^2 \sin x - 2 \int x \sin x \, dx$$

Again by parts method

$$= x^2 \sin x - 2 \left[x \int \sin x \, dx - \int \left\{ \frac{d}{dx} x \int \sin x \, dx \right\} dx \right]$$

$$= x^2 \sin x - 2 \left[x(-\cos x) - \int \{1 \times -\cos x\} dx \right]$$

$$= x^2 \sin x - 2 \left[-x \cos x + \int \cos x \, dx \right]$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + c$$

(ii) $\int x^2 e^x \, dx = x^2 \int e^x \, dx - \int \left\{ \frac{d}{dx} x^2 \int e^x \, dx \right\} dx$

$$= x^2 e^x - \int \{2x e^x\} dx$$

$$= x^2 e^x - 2 \int x e^x \, dx$$

Again by parts method

$$= x^2 e^x - 2 \left[x \int e^x \, dx - \int \left\{ \frac{d}{dx} x \int e^x \, dx \right\} dx \right]$$

$$= x^2 e^x - 2 \left[x e^x - \int (1 e^x) dx \right]$$

$$= x^2 e^x - 2 x e^x + e^x + c$$

EXERCISE -VI

1. $\int x \cos x \, dx$
2. $\int \log(x+1) dx$
3. $\int x e^x \, dx$
4. $\int x^2 \log x \, dx$
5. $\int x \cot^2 x \, dx$
6. $\int x^2 \cos 2x dx$
7. $\int x^2 e^{-x} \, dx$
8. $\int x \operatorname{cosec}^2 x \, dx$
9. $\int \frac{6x-1}{3x^2-x+4} dx$
10. $\int \frac{1}{x \log x} dx$
11. $\int \frac{\sec^2 x}{\tan x} dx$
12. $\int \frac{2x}{e^{x^2+1}} dx$

ANSWERS

1. $x \sin x + \cos x + c$
2. $x \log |(x+1)| - x + \log |(x+1)| + c$
3. $x e^x - e^x + c$
4. $\frac{x^3}{3} \log |x| - \frac{1}{9} x^3 + c$
5. $-x \cot x + \log |\sin x| - \frac{x^2}{2} + c$
6. $\frac{x^2}{2} \sin 2x + \frac{x}{2} \cos 2x - \frac{\sin 2x}{4} + c$
7. $-x^2 e^{-x} - 2x e^{-x} - 2e^{-x} + c$
8. $-x \cot x + \log \sin x + c$
9. $\log |3x^2 - x + 4| + c$
10. $\log |\log x| + c$
11. $\log |\tan x| + c$
12. $-e^{-(x^2+1)} + c$

3.3 (a) Evaluation of definite integrals with given limits

Introduction: If $f(x)$ is a continuous function defined on closed interval $[a, b]$ and $F(x)$ is the integral of $f(x)$ i.e.

$$\int f(x) dx = F(x)$$

then definite integral of $f(x)$ in closed interval $[a, b]$ is

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a)$$

where a is lower limit, b is upper limit.

Example 12. Evaluate (i) $\int_1^2 x^2 dx$ (ii) $\int_0^{\pi/4} \sin x dx$ (iii) $\int_0^1 e^x dx$

(iv) $\int_4^9 \sqrt{x} dx$ (v) $\int_1^2 \frac{1}{x} dx$

Sol : (i) $\int_1^2 x^2 dx = \left[\frac{x^3}{3} \right]_1^2 = \left(\frac{2^3}{3} - \frac{1^3}{3} \right) = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$.

(ii) $\int_0^{\pi/4} \sin x dx = [-\cos x]_0^{\pi/4} = \left(-\cos \frac{\pi}{4} \right) - (-\cos 0) = \left(-\frac{1}{\sqrt{2}} + 1 \right)$

(iii) $\int_0^1 e^x dx = \left[\frac{e^x}{1} \right]_0^1 = (e^1 - e^0) = (e - 1)$

(iv) $\int_4^9 \sqrt{x} dx = \left[\frac{x^{3/2}}{3/2} \right]_4^9 = \frac{2}{3} [x^{3/2}]_4^9 = \frac{2}{3} [9^{3/2} - 4^{3/2}]$
 $= \frac{2}{3} [27 - 8] = 2 \times \frac{19}{3} = \frac{38}{3}$

(v) $\int_1^2 \frac{1}{x} dx = [\log x]_1^2 = \log 2 - \log 1 = \log 2$ as $(\log 1 = 0)$

Example 13. Evaluate (i) $\int_0^1 \frac{dx}{1+x^2}$ (ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (iii) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol : (i) $\int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0$

$$= \tan^{-1} \tan \frac{\pi}{4} - \tan^{-1} \tan 0 = \left(\frac{\pi}{4} - 0 \right) = \frac{\pi}{4}.$$

(ii) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}} = [\sec^{-1} x]_1^2$

$$= \sec^{-1} 2 - \sec^{-1} 1$$

$$= \sec^{-1} \sec \frac{\pi}{3} - \sec^{-1} \sec 0$$

$$= \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

EXERCISE -VII

1. The value of the integral $\int_0^{\pi/4} \tan x \, dx$.

- a. $\log 2$ b. $\frac{1}{2}$ c. $\log \sqrt{2}$ d. 2

2. The integral $\int_{-2}^2 e^x \, dx$

- a. $e^2 - e^{-2}$ b. $e^{-2} - e^2$ c. $e^{-2} + e^2$ d. $2e^2$

3. Evaluate : (i) $\int_2^3 \frac{dx}{x}$ (ii) $\int_{-5}^5 x \, dx$ (iii) $\int_1^2 \frac{dx}{1+3x}$

(iv) $\int_2^3 (x^2 + 1) dx$ (v) $\int_{-2}^2 e^{-x} \, dx$ (vi) $\int_0^{\pi/3} \cos x \, dx$

(vii) $\int_0^2 \frac{dx}{4+3x}$ (viii) $\int_0^2 \frac{dx}{x^2+4}$ (ix) $\int_0^{\pi/4} \tan x \, dx$

ANSWERS

1. (c) 2. (a)

3. (i) $\log \frac{3}{2}$ (ii) 0 (iii) $\frac{1}{3} \log \frac{7}{4}$ (iv) $\frac{22}{3}$ (v) $-e^{-2} + e^2$
 (vi) $\frac{\sqrt{3}}{2}$ (vii) $\frac{1}{3} \log \frac{5}{2}$ (viii) $\frac{\pi}{8}$ (ix) $\log \sqrt{2}$.

(b) **Evaluation of** $\int_0^{\pi/2} \sin^n x \, dx$, $\int_0^{\pi/2} \cos^n x \, dx$, $\int_0^{\pi/2} \sin^m x \cos^n x \, dx$

using formulae without proof (m and n being positive integers only) using pre-existing mathematical models

$$(i) \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{(n-1)(n-3)\dots 3.1}{n(n-2)(n-4)\dots 4.2} \times \frac{\pi}{2} & \text{if } n = \text{even} \\ \frac{(n-1)(n-3)\dots 4.2}{n(n-2)\dots 5.3} & \text{if } n = \text{odd} \end{cases}$$

$$(ii) \int_0^{\pi/2} \cos^n x \, dx = \begin{cases} \frac{(n-1)(n-3)\dots 3.1}{n(n-2)(n-4)\dots 4.2} \times \frac{\pi}{2} & \text{if } n = \text{even} \\ \frac{(n-1)(n-3)\dots 4.2}{n(n-2)\dots 5.3} & \text{if } n = \text{odd} \end{cases}$$

$$(iii) \int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{(n-1)(n-3)\dots(m-1)(m-3)\dots}{(n+m)(n+m-2)(n+m-4)\dots} \times \frac{\pi}{2}$$

(if n and m both are even)

$$(iv) \int_0^{\pi/2} \sin^n x \cos^m x \, dx = \frac{(n-1)(n-3)\dots(m-1)(m-3)\dots}{(n+m)(n+m-2)(n+m-4)\dots}$$

(if either n or m or both are odd)

Note : Above formulae are called 'Reduction Formulae'

Example 14. Evaluate (i) $\int_0^{\pi/2} \sin^5 x \, dx$ (ii) $\int_0^{\pi/2} \cos^4 x \, dx$ (iii) $\int_0^{\pi/2} \sin^3 x \cos^2 x \, dx$

Sol : (i) $\int_0^{\pi/2} \sin^5 x \, dx = \frac{4 \times 2}{5 \times 3 \times 1} = \frac{8}{15}$

$$(ii) \quad \int_0^{\pi/2} \cos^4 x \, dx = \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = \frac{3\pi}{16}$$

Here $n = 4$ (even)

$$(iii) \quad \int_0^{\pi/2} \sin^3 x \cos^2 x \, dx = \frac{2 \times 1}{5 \times 3 \times 1} = \frac{2}{15}$$

EXERCISE-VIII

Evaluate the integral :

$$1. \quad \int_0^{\pi/2} \sin^8 x \, dx$$

$$2. \quad \int_0^{\pi/2} \cos^5 x \, dx$$

$$3. \quad \int_0^{\pi/2} \sin^4 x \, dx$$

$$4. \quad \int_0^{\pi/2} \cos^6 x \, dx$$

$$5. \quad \int_0^{\pi/2} \cos^3 x \sin^4 x \, dx$$

$$6. \quad \int_0^{\pi/2} \cos^6 x \sin^4 x \, dx$$

$$7. \text{ Evaluate } I = \frac{\int_0^{\pi/2} \sin^3 x \cos^5 x \, dx}{\int_0^{\pi/2} \cos^4 x \, dx}$$

ANSWERS

$$1. \quad \frac{35\pi}{256} \quad 2. \quad \frac{8}{15} \quad 3. \quad \frac{3\pi}{16} \quad 4. \quad \frac{5\pi}{96} \quad 5. \quad \frac{2}{35}$$

$$6. \quad \frac{3\pi}{512} \quad 7. \quad \frac{2}{9\pi}$$

* * * * *

UNIT- IV

Application of Integration, Numerical Integration and Differential Equations

4.1 Applications of integration: for evaluation of area under a curve and axes (Simple problems):

There are many applications that requires integration techniques like Area under the curve, volumes of solid generated by revolving the curve about axes, velocity, acceleration, displacement, work done, average value of function. But in this chapter, we will be taking a look at couple of applications of integrals.

- (i) If $y = f(x)$ is any function of x , then

Area under the curve is $\int_{x=a}^{x=b} y \, dx$

- (ii) If $v = f(t)$ is velocity of particle at time t then,

Displacement $S = \int_{t=t_1}^{t=t_2} v \, dt$

Example 1. Find the area under the curve $y = 1 + 2x^3$, when $0 \leq x \leq 2$.

Sol : Given $y = 1 + 2x^3$, $a = 0$, $b = 2$.

$$\text{Area} = \int_a^b y \, dx = \int_0^2 (1 + 2x^3) \, dx$$

$$= \left[x + \frac{x^4}{2} \right]_0^2 = (2 + 8) - (0 + 0) = 10 \text{ units}$$

Example 2. Find area under the curve of $y = \sin x$, when $0 \leq x \leq \frac{\pi}{2}$.

Sol : Equation of curve $y = \sin x$, $a = 0$, $b = \frac{\pi}{2}$

$$\begin{aligned}
 \text{Area} &= \int_a^b y \, dx = \int_0^{\pi/2} \sin x \, dx = [-\cos x]_0^{\pi/2} \\
 &= [-\cos \frac{\pi}{2}] - [-\cos 0] \\
 &= [-0] + 1 = 1 \quad \text{Square units}
 \end{aligned}$$

EXERCISE-I

- Find the area under the curve $y = 4x^2$, when $0 \leq x \leq 3$.
- Find the area under the curve $y = \frac{1}{x}$, when $2 \leq x \leq 4$.
- Find the area under the curve $y = \cos x$, when $0 \leq x, \frac{\pi}{4}$.
- Find the area under the curve $y = e^{2x}$, when $0 \leq x \leq 1$.
- Find area under the curve $y = \sqrt{2x+3}$, when $3 \leq x \leq 11$.

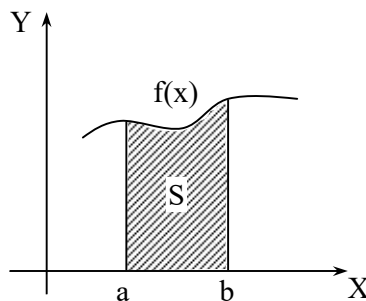
ANSWERS

- 36
- $\log 2$
- $\frac{1}{\sqrt{2}}$
- $\frac{e^2 - 1}{2}$
- $\frac{98}{3}$

4.2 Numerical integration by Trapezoidal Rule and Simpson's 1/3rd Rule using pre-existing mathematical models:

Numerical integration is an approximate solution to a definite integral $\int_a^b f(x) \, dx$. In this chapter we are taking two methods to

find approximate value of definite integral, $\int_a^b f(x) \, dx$ as area under the curve when $a \leq x \leq b$.



(i) Trapezoidal Rule

(ii) Simpson's Rule

(i) Trapezoidal Rule : If $y = f(x)$ be the equation of curve, then approximate area under the curve $y = f(x)$, when $a \leq x \leq b$ is

$$\int_a^b y \, dx = \frac{h}{2} [(first\ ordinate + last\ ordinate) + 2 (\text{sum of remaining ordinates})]$$

Here $h = \frac{b-a}{n}$

$a \rightarrow$ lower limit of x

$b \rightarrow$ upper limit of x

$n \rightarrow$ Number of intervals

* Ordinates are always one more than the number of intervals.

In mathematics, and more specifically in Number analysis, the trapezoidal rule (also known as trapezoid rule or trapezium rule) is a technique, for approximating the definite integral $\int_a^b f(x) \, dx$. The trapezoidal rule works by approximating the region under the graph of function $f(x)$ as a trapezoid and calculating its area.

Example 3. Find the approximate area under the curve using trapezoidal rule, determined by the data given below

x	0	1	2	3	4	5
y	0	2.5	3	4.5	5	7.5

Sol : Here equation of curve not given so let $y = f(x)$

Lower limit of x , i.e. $a = 0$

Upper limit of x , i.e. $b = 5$

Gap between values of x i.e. $h = 1$

Number of ordinates in table are = 6

So by trapezoidal Rule

$$\int_0^5 y \, dx = \frac{h}{2} [(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$$y_1 = 0, y_2 = 2.5, y_3 = 3, y_4 = 4.5, y_5 = 5, y_6 = 7.5$$

$$= \frac{1}{2} [(0 + 7.5) + 2(2.5 + 3 + 4.5 + 5)]$$

$$= \frac{1}{2} [7.5 + 30] = 18.75 \text{ square units}$$

Example 4. Using trapezoidal rule, evaluate $\int_0^{2.5} (1+x) \, dx$ by taking six ordinates.

Sol : Given $y = 1 + x$, $a = 0$, $b = 2.5$

as ordinates are 6 $\Rightarrow n = 5$

$$h = \frac{b-a}{n} = \frac{2.5-0}{5} = 0.5$$

x	0	0.5	1	1.5	2	2.5
y = 1 + x	1	1.5	2	2.5	3	3.5

Here $y_1 = 1, y_2 = 1.5, y_3 = 2, y_4 = 2.5, y_5 = 3, y_6 = 3.5$.

By trapezoidal Rule

$$\int_0^{2.5} (1+x) \, dx = \frac{h}{2} [(y_1 + y_6) + 2(y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{0.5}{2} [(1 + 3.5) + 2(1.5 + 2 + 2.5 + 3)]$$

$$= \frac{1}{4} [4.5 + 18] = \frac{22.5}{4} = 5.6 \text{ square units}$$

EXERCISE -II

- Using trapezoidal rule, evaluate $\int_0^2 \sqrt{9-x^2} dx$ by taking 4 equal intervals.
- Using trapezoidal Rule, Find the approximate area under the curve $y = x^2 + 1$, when $0 \leq x \leq 6$ by taking 7 ordinates.
- A curve is drawn to pass through the points given below

x	2	2.2	2.4	2.6	2.8	3.0
y	3	3.4	3.7	3.9	4	3.2

Find the approximate area bounded by the curve, x axis and the lines $x = 2$ and $x = 3$.

- Apply Trapezoidal rule to evaluate $\int_4^8 \frac{1}{x+2}$ by taking 4 equal intervals.

ANSWERS

- 5.4
- 79
- 3.62
- 0.5123

Simpson's $\frac{1}{3}^{rd}$ Rule : In this rule, the graph of curve $y = f(x)$ is divided into $2n$ (Even) intervals.

Let $y = f(x)$ be the equation of curve. When $a \leq x \leq b$, then approximate area under the curve is

$$\int_a^b y dx = \frac{h}{3} [(first\ ordinate + last\ ordinate) + 2(sum\ of\ remaining\ odd\ ordinates) + 4(sum\ of\ remaining\ even\ ordinates)]$$

$$\text{where } h = \frac{b-a}{n}$$

$a \rightarrow$ lower limit of x

$b \rightarrow$ upper limit of x

$n \rightarrow$ number of intervals

Note : Simpson's Rule is valid when total number of ordinates are odd.

Example 5. Calculate by Simpson's rule an approximate value of $\int_2^8 (x^3 + 3)dx$ by taking 7 ordinates.

Sol : Given $y = x^3 + 3$, $a = 2$, $b = 8$, $n = 6$

$$\therefore h = \frac{b-a}{n} = \frac{8-2}{6} = 1$$

x	2	3	4	5	6	7	8
$y = x^3 + 3$	11	30	67	128	219	347	515

Here $y_1 = 11$, $y_2 = 30$, $y_3 = 67$, $y_4 = 128$, $y_5 = 219$, $y_6 = 347$, $y_7 = 515$.

By Simpson's rule

$$\begin{aligned} \int_2^8 (x^3 + 3)dx &= \frac{h}{3} [(y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)] \\ &= \frac{1}{3} [(11 + 515) + 2(67 + 219) + 4(30 + 128 + 347)] \\ &= \frac{1}{3} [526 + 572 + 2020] = \frac{1}{3} [3118] \\ &= 1039.3 \text{ square units.} \end{aligned}$$

Example 6. Evaluate $\int_0^4 e^x dx$ by Simpson's rule, when $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$.

Sol. Given $y = e^x$, $a = 0$, $b = 4$, $h = 1$ (As values of x varies in gap of 1).

x	0	1	2	3	4
$Y = e^x$	$e^0 = 1$	$e^1 = 2.72$	$e^2 = 7.39$	$e^3 = 20.09$	$e^4 = 54.6$

$$y_1 = 1, y_2 = 2.72, y_3 = 7.39, y_4 = 20.09, y_5 = 54.6$$

By Simpson's Rule

$$\begin{aligned}
 \int_0^4 e^x dx &= \frac{h}{3} [(y_1 + y_5) + 2y_3 + 4(y_2 + y_4)] \\
 &= \frac{1}{3} [(1 + 54.6) + 2 \times 7.39 + 4(2.72 + 20.09)] \\
 &= \frac{1}{3} [55.6 + 14.78 + 91.24] = 53.87.
 \end{aligned}$$

Example 7. Using Simpson's rule, evaluate

$$\int_0^{12} x^2 dx$$

by taking 6 equal intervals.

Sol. Here $a = 0, b = 12, n = 6$ and $y = f(x) = x^2$

$$\text{Let } h = \frac{12-0}{6} = \frac{12}{6} = 2$$

$$x_1 = 0$$

$$y_1 = x_1^2 = 0$$

$$x_2 = 2$$

$$y_2 = x_2^2 = 4$$

$$x_3 = 4$$

$$y_3 = x_3^2 = 16$$

$$x_4 = 6$$

$$y_4 = x_4^2 = 36$$

$$x_5 = 8$$

$$y_5 = x_5^2 = 64$$

$$x_6 = 10$$

$$y_6 = x_6^2 = 100$$

$$x_7 = 12$$

$$y_7 = x_7^2 = 144$$

Therefore,

$$\begin{aligned}
 \int_0^{12} x^2 dx &= \frac{h}{3} [(y_1 + y_7) + 2(y_3 + y_5) + 4(y_2 + y_4 + y_6)] \\
 &= \frac{2}{3} [(0 + 144) + 2(16 + 64) + 4(4 + 36 + 100)] \\
 &= \frac{2}{3} [144 + 2(80) + 4(140)] \\
 &= \frac{2}{3} [864] = 576
 \end{aligned}$$

EXERCISE - III

1. Calculate by Simpson's rule an approximate value of $\int_0^1 \frac{dx}{1+x}$ by taking 10 equal interval.
2. Apply Simpson's rule to evaluate approximate value of $\int_4^8 \frac{dx}{x-3}$ by taking four equal intervals.

3. Find approximate area under the curve $y = (1 + x^2)$ when $0 \leq x \leq 4$ using Simpson's Rule by taking five ordinates.
4. Use Simpson's rule to approximate the area under the curve determined by the data given below

x	1	2	3	4	5	6	7
y	15	27	30	33	20	19	4

ANSWERS

1. .6931 2. 1.62 3. 25.33 4. 145

Differential Equations

4.3 Definition, order, degree, Type of differential Equations, linearity, Formulation of ordinary differential equation (up to 1st order), solution of ODE (1st order) by variable separation method

Differential Equations: In engineering problems, mathematical models are made to represent certain problems. These mathematical models involve variables and derivatives of unknown functions. Such equations form the differential equation. Thus, a Differential Equation is defined as an equation in which y and its derivatives exist in addition to the independent variable x .

For example; (i) $\frac{dy^2}{dx^2} + 5y = 0$ (ii) $\frac{dy}{dx} + 3xy = 0$ (iii) $\left(\frac{d^2y}{dx^2}\right)^2 = 1 + \frac{dy}{dx}$

In this chapter we will study only about ordinary differential equation.

Ordinary differential Equation: A differential equation in which dependent variable ' y ' depends on only one variable ' x ' i.e., $y = f(x)$. e.g.

$$\frac{dy}{dx} + y = 0$$

Partial differential Equation : A differential equation in which dependent variable ' z ' depends on more than one variable say x and y i.e., $z = f(x, y)$, For example,

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

Order of differential equation : It is the order of the highest derivative appearing in the differential equation.

Example 1. Find the order (i) $\frac{d^2 y}{dx^2} + 5y = 0$ (ii) $\frac{d^3 y}{dx^3} + 2\left(\frac{d^2 y}{dx^2}\right)^2 + 3y = 0$

Sol. (i) order is 2 (ii) order is 3

Degree of Differential Equation: It is the power of highest derivative appearing in a differential equation when it is free from radicals i.e. as far as derivatives are concerned fractional and negative powers should be removed.

Example 2. Write the order and Degree of following differential equations

(i) $\left(\frac{d^2 y}{dx^2}\right)^3 + y = 0$

(ii) $\frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$

(iii) $\left(\frac{d^2 y}{dx^2}\right)^2 = \frac{1}{1 + \frac{d^2 y}{dx^2}}$

Solution : (i) $\left(\frac{d^2 y}{dx^2}\right)^3 + y = 0$

Order $\rightarrow 2$

Degree $\rightarrow 3$

(ii) $\frac{d^2 y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^3}$

Squaring both sides

$$\left(\frac{d^2 y}{dx^2}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^3$$

Order $\rightarrow 2$

Degree $\rightarrow 2$

$$(iii) \left(\frac{d^2 y}{dx^2} \right)^2 = \frac{1}{1 + \frac{d^2 y}{dx^2}}$$

Cross multiply both sides

$$\left(\frac{d^2 y}{dx^2} \right)^2 \times \left[1 + \frac{d^2 y}{dx^2} \right] = 1$$

$$\left(\frac{d^2 y}{dx^2} \right)^2 + \left(\frac{d^2 y}{dx^2} \right)^3 = 1$$

Order – 2,

Degree 3

Linear Differential Equation : A differential equation in which dependent variable 'y' and its derivatives has power one and they are not multiplied together. When $y = f(x)$

Example : $\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} + 6 \frac{dy}{dx} + y = 0$

First Order Linear Differential Equation

A Differential Equation of the form

$$\frac{dy}{dx} + P(x)y = 0 \quad \text{where } y = f(x)$$

is called first order linear differential equation.

EXERCISE -I

1. Define the following with Examples

(i) Differential equation

(ii) Order of Differential Equation

(iii) Degree of Differential Equation

2. Find the order and degree of the differential equation $\frac{d^3 y}{dx^3} + \left(\frac{d^2 y}{dx^2} \right)^3 + y = 0$

a. Order= 3, degree= 3

c. Order= 2, degree= 3

b. Order= 3, degree= 2

d. Order= 3, degree= 1

3. Find the degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 = \frac{1}{1 + \frac{d^2y}{dx^2}}$

- a. 1 b. 2 c. 4 d. None of these

4. Find the degree of the differential equation $(x+1) = \left(\frac{d^2y}{dx^2}\right)^{\frac{2}{3}}$

- a. $\frac{2}{3}$ b. 3 c. 2 d. 1

5. Determine the order and degree of the following differential equation. State whether the equations is linear or non linear

(i) $\left(\frac{d^3y}{dx^3}\right)^2 + 7\left(\frac{d^2y}{dx^2}\right)^{10} + 9\frac{dy}{dx} + 7y = 0$

(ii) $\frac{dy}{dx} + \cos x = 0$

(iii) $x\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y = 0$

(iv) $\frac{d^2y}{dx^2} + \log x \frac{dy}{dx} + y^2 = 0$

(v) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + 5y = 0$

(vi) $\left(\frac{d^2y}{dx^2}\right)^{\frac{1}{3}} = \left(\frac{dy}{dx}\right)^{\frac{1}{2}}$

(vii) $\frac{d^4y}{dx^4} + 3\frac{dy}{dx} = 1$

(viii) $\left[1 + \left(\frac{d^2y}{dx^2}\right)^2\right]^{\frac{1}{3}} = \frac{d^3y}{dx^3}$

ANSWERS

- 2.(d) 3. (d) 4. (c)
5. (i) Order - 3 degree - 2 non linear
- (ii) Order - 1 degree - 1 Linear
- (iii) Order - 2 degree - 1 non linear
- (iv) Order - 2 degree - 1 non linear
- (v) Order - 3 degree - 1 non linear
- (vi) Order - 2 degree - 2 non linear
- (vii) Order - 4 degree - 1 linear
- (viii) Order – 3 degree - 3 non linear

Solution of a Differential Equation: The solution of differential equation is the relation between the variables involves in differential equation which satisfies the given differential equation.

In this chapter we will study the solution of first order differential equation by variable separable method.

Steps :

1. Separate the variables in given differential equation keeping in mind that dy and dx should be in numerator.
2. Now put the sign of $\int$ on both sides.
3. Integrate both sides separately by adding a constant on one side.
4. This will give us general sol of Differential equation.

Example 3. Find the general solution of differential equation $\frac{dy}{dx} = \sin x$.

Sol : Given $\frac{dy}{dx} = \sin x$

Separate the variables, we get

$$dy = \sin x \, dx$$

Put the sign of $\int$ on both sides

$$\int dy = \int \sin x \, dx$$

$$y = -\cos x + C$$

Example 4. Find the general solution of differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$.

Sol : Separate the variables $\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides, we have

$$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$$\tan^{-1}y = \tan^{-1}x + c$$

Example 5. Find the solution of differential equation :

(i) $\frac{dy}{dx} = \frac{x}{y}$

(ii) $\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$

(iii) $x dy - y dx = 0$

(iv) $\frac{dy}{dx} = \frac{1+y}{1+2x}$

Sol : (i) $\frac{dy}{dx} = \frac{x}{y}$

$$\Rightarrow y dy = x dx$$

$$\Rightarrow \int y dy = \int x dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$(ii) \quad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow dy = \frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \int dy = \int \frac{dx}{\sqrt{1+x^2}}$$

$$\Rightarrow y = \log |x + \sqrt{1+x^2}| + c$$

$$(iii) \quad x \, dy - y \, dx = 0$$

$$\Rightarrow x \, dy = y \, dx$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + c$$

$$(iv) \quad \frac{dy}{dx} = \frac{1+y}{1+2x}$$

$$\Rightarrow \frac{dy}{1+y} = \frac{dx}{1+2x}$$

$$\Rightarrow \int \frac{dy}{1+y} = \int \frac{dx}{1+2x}$$

$$\Rightarrow \log(1+y) = \frac{\log(1+2x)}{2} + c$$

EXERCISE-II

1. Solve the differential equation $\frac{dy}{dx} = 1 + x + y + xy$
 - a. $y = x^2 + xy + x + c$
 - b. $\log(1 + x) = y + y^2 + c$
 - c. $\log(1 + y) = x + \frac{x^2}{2} + c$
 - d. $(1 + x)y = c$
2. The general sol of the differential equation $\frac{dy}{dx} = \frac{2-y}{x+1}$ is
 - a. $\log\left[\frac{2-y}{x+1}\right] = c$
 - b. $\log[(2-y)(x+1)] = c$
 - c. $\log(x+1)y = x + c$
 - d. None of these
3. The general sol of the differential equation $\frac{dy}{dx} = e^x \cdot e^{-y}$ is
 - a. $e^y = e^x + c$
 - b. $e^x + e^{-y} = c$
 - c. $e^{-y} = e^x + c$
 - d. $e^x e^y = c$
4. Solve the differential equations
 - (i) $\frac{dy}{dx} = \frac{x+2}{y}$
 - (ii) $\frac{dy}{dx} = \frac{2-y}{x+1}$
 - (iii) $\frac{dy}{dx} = y \tan 2x$
 - (iv) $\frac{dy}{dx} = \sqrt{\frac{1+y^2}{1+x^2}}$
 - (v) $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$
 - (vi) $(1-x)ydx + (1+y)x dy = 0$
5. Solve the differential equation $\frac{dy}{dx} = 1 - x + y - xy$.
6. Solve the differential equation $\sin x \cos y dx + \cos x \sin y dy = 0$
7. Solve the differential equation $\frac{dy}{dx} = 2e^x y^3$.

ANSWERS

1. (c)
2. (b)
3. (a)
4. (i) $\frac{y^2}{2} = \frac{x^2}{2} + 2x + c$
- (ii) $\frac{\log(2-y)}{-1} = \log(x+1) + c$

$$(iii) \log y = \frac{\log \sec 2x}{2} + c$$

$$(iv) \log |y + \sqrt{1+y^2}| = \log |x + \sqrt{1+x^2}| + c$$

$$(v) e^y = e^x + \frac{x^3}{3} + c$$

$$(vi) \log xy = x - y + c$$

$$5. \log(1+y) = x - \frac{x^2}{2} + c$$

$$6. \log \sec x + \log \sec y = c$$

$$7. -\frac{y^{-2}}{2} = 2e^x + c$$

* * * * *

UNIT- V

STATISTICS AND SOFTWARE

Statistics: Statistics is a branch of mathematics dealing with collection of data, analysis, interpretation, presentation and organization of data. In applying statistics into a problem, to for Example a scientific, industrial or social problem, it is conventional to begin with statistical population or a statistical model process to be studied in that problem.

Here in this chapter, we study about some measures of central tendency which are a central or typical values for a probability distribution. In other words, measures of central tendency are often called averages. Before we study about measures of central tendency, we must know about type of data.

Type of Data

a) Raw data : Data collected from source without any processing.

For example : The marks of ten students in math subject are 10,12,15,25,30,25,15,13,20,25

b) Discrete frequency distribution (Ungrouped data): The data which have not been divided into groups and presented with frequency (no. of times appeared) of data.

For example : The marks of ten students in math subject are given by

Marks(x):	10	12	13	15	20	25	30
Frequency(f):	1	1	1	2	1	3	1

c) Continuous frequency distribution (Grouped data): The data which have been divided into continuous group and presented with frequency (no. of times appeared) of individual group.

For example : The marks of students in math subject in a class are given by:

Marks class	10-20	20-30	30-40	40-50
Frequency	7	10	15	8

5.1 MEASURE OF CENTRAL TENDENCY

Measure of Central tendency are also classed as summary of statistics. In general, the mean, median and mode are all valid measure of central tendency.

Mean: It is average value of given data.

For Raw data ; **Mean** $\bar{x} = \frac{\sum x_i}{n}$; where x_i is value of individual data

For Ungrouped and Grouped data; **Mean** $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$; where x_i is the value of individual data and f_i is corresponding frequency of value x_i .

Median : It is middle value of arranged data.

The class corresponding to cumulative frequency just greater than $\frac{N}{2}$ is called the Median class and value of median is calculated by formula

For Grouped data; **Median** $= l + \frac{\left(\frac{n}{2} - c\right)}{f} * h$

l = lower limit of class

$n = \sum f_i$,

c = Cumulative frequency preceding to the median class cumulative frequency

h = length of interval,

f = frequency of median class

Mode : It is the term which appears maximum number of times in the given data.

OR

The term which has highest frequency in given data.

For Grouped data; **Mode** $= l + \frac{(f - f_1)}{2f - f_1 - f_2} * h$

Modal class = class which has maximum frequency value

where l = lower limit of modal class,

h = length of interval,

f = frequency of modal class

f_1 = frequency of preceding class to modal class

f_2 = frequency of succeeding class to modal class

Example 1. Calculate mean for the following data.

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol : The given data is 21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Total No. of observations = 10

Sum of observations = $21 + 23 + 25 + 28 + 30 + 32 + 46 + 38 + 48 + 46 = 337$

$$\text{Mean} = \frac{\text{Sum of observations}}{\text{Total No. of observations}} = \frac{337}{10} = 33.7$$

Example 2. Find the arithmetic mean of first 10 natural numbers.

Sol : First 10 natural numbers are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10

Sum of numbers = $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

No. of terms = 10

$$\text{Mean} = \frac{\text{sum of numbers}}{\text{No. of terms}} = \frac{55}{10} = 5.5$$

Example 3. Find the median of the daily wages of ten workers.

(Rs.) : 20, 25, 17, 18, 8, 15, 22, 11, 9, 14

Sol : Arranging the data in ascending order, we have

8, 9, 11, 14, 15, 17, 18, 20, 22, 25

Since there are 10 observations, therefore median is the arithmetic mean of $\left(\frac{10}{2}\right)^{th}$ and

$$\left(\frac{10}{2} + 1\right)^{th} \text{ observations, so median} = \frac{15 + 17}{2} = 16$$

Example 4. The following are the marks of 9 students in a class. Find the median.

34, 32, 48, 38, 24, 30, 27, 21, 35

Sol : Arranging the data in ascending order, we have

21, 24, 27, 30, 32, 34, 35, 38, 48

Since there are total 9 number of terms, which is odd. Therefore, median is the value of

$$\left(\frac{9+1}{2}\right)^{th} \text{ observation i.e. 32}$$

Example 5. Find the mode from the following data:

110, 120, 130, 120, 110, 140, 130, 120, 140, 120

Sol : Arranging the data in the form of a frequency table, we have

Value:	110	120	130	140
Frequency :	2	4	2	2

Since the value 120 occurs the maximum number of times. Hence the mode value is 120

Example 6. The arithmetic mean of 7, 9, 5, 2, 4, 8, x is given to be 7. Find x.

$$\text{Sol : } \bar{x} = \frac{7+9+5+2+4+8+x}{7}$$

$$\text{but } \bar{x} = 7$$

$$7 = \frac{35+x}{7}$$

$$\Rightarrow 49 = 35 + x$$

$$\Rightarrow x = 49 - 35 = 14$$

Example 7. Calculate range for the following data

19, 25, 36, 72, 51, 43, 28

Sol : Maximum value of given data = 72

Minimum value of given data = 19

$$\begin{aligned}\text{Range of data} &= \text{Maximum value} - \text{Minimum value} \\ &= 72 - 19 = 53\end{aligned}$$

Example 8. Calculate arithmetic mean for the following :

Income (in Rs.)	:	500	520	550	600	800	1000
No. of Employees	:	4	10	6	5	3	2

Income (in Rs.)	No. of Employees	
(x_i)	(f_i)	$(f_i x_i)$
500	4	2000
520	10	5200
550	6	3300
600	5	3000
800	3	2400
1000	2	2000
$\Sigma f_i = 30$		$\Sigma x_i f_i = 17900$

$$\Rightarrow \bar{x} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{17900}{30} = 596.67$$

Example 9. Calculate median for the following data

x_i	:	1	2	3	4	5	6	7	8	9
f_i	:	8	10	11	16	20	25	15	9	6

Sol :

x_i	f_i	C.f
1	8	8
2	10	18
3	11	29
4	16	45
5	20	65
6	25	90
7	15	105
8	9	114
9	6	120
Σf_i		$=N = 120$

Here $N = 120$

$$\text{Now } \frac{N}{2} = \frac{120}{2} = 60$$

We find that the cumulative frequency just greater than $\frac{N}{2}$, is 65 and the value of x corresponding to 65 is '5'. Therefore median is '5'.

Example 10. Calculate mode for the following data:

x_i	91	92	96	97	101	103	108
f_i	3	2	3	2	5	3	3

Sol : As we know that mode is the highest frequency value and highest frequency is 5 and corresponding value is 101.

So mode value is 101

Example 11. Calculate mean for the following frequency distribution

Class interval :	0-8	8-16	16-24	24-32	32-40	40-48
Frequency :	8	7	16	24	15	7

Sol :

Class interval	Frequency(f_i)	Mid Value (x_i)	$f_i x_i$
0-8	8	4	32
8-16	7	12	84
16-24	16	20	320

24-32	24	28	672
32-40	15	36	540
40-48	7	44	308
	$\Sigma f_i = 77$		$\Sigma x_i f_i = 1956$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1956}{77} = 25.40$$

Example 12. Calculate the median from the following distribution

Class : 5-10 10-15 15-20 20-25 25-30 30-35 35-40 40-45

Fre. : 5 6 15 10 5 4 2 2

Sol :

Class	f_i	C.f
5-10	5	5
10-15	6	11
15-20	15	26
20-25	10	36
25-30	5	41
30-35	4	45
35-40	2	47
40-45	2	49
	$\Sigma f_i = N = 49$	

We have $N = 49$

$$\text{Now } \frac{N}{2} = \frac{49}{2} = 24.5$$

commutative frequency just

greater than $\frac{N}{2}$, is 24.5 and

Corresponding class is 15-20

Thus, class 15-20 is the median

class such that $l = 15$, $f = 15$

c.f. = 11, $h = 5$

$$\Rightarrow \text{Median} = l + \frac{\left(\frac{N}{2} - C\right)}{f} \times h = 15 + \frac{(24.5 - 11)}{15} \times 5 = 15 + \frac{13.5}{3} = 15 + 4.5 = 19.5$$

Example 13. Calculate mode from the following data

Rent(in Rs.) : 20-40 40-60 60-80 80-100 100-120 120-140 140-160

No. of House : 6 9 11 14 20 15 10

Sol : By observation, we find that the highest frequency is 20

Hence 100-120 is the modal class.

$$\text{Mode} = l + \frac{(f - f_1)}{(2f - f_1 - f_2)} \times h$$

where $l = 100$, $f = 20$, $f_1 = 14$, $f_2 = 15$, $h = 20$

$$\begin{aligned} \Rightarrow \text{Mode} &= 100 + \frac{(20 - 14)}{2(20) - 14 - 15} \times 20 \\ &= 100 + \frac{6 \times 20}{11} = 100 + \frac{120}{11} = 100 + 10.91 = 110.91 \end{aligned}$$

5.2 MEASURE OF DISPERSION

Mean Deviation : The mean deviation is the first measure of dispersion. It is the average of absolute differences between each value in a set of value, and the average value of all the values of that set. Mean Deviation is calculated either from Mean or median. Here, we will study Mean Deviation about Mean

1. Mean deviation about Mean

(a) For Raw data

$$\text{Mean Deviation about Mean} = \frac{\sum |x_i - \bar{x}|}{N}$$

Where $\Sigma \rightarrow$ Represents summation

$N \rightarrow$ Number of terms or observations

$x_i \rightarrow$ Value of individual data

$$\text{Mean } \bar{x} = \frac{\sum x_i}{N}$$

(b) For Ungrouped and Grouped data

$$\text{Mean Deviation about Mean} = \frac{\sum f_i |x_i - \bar{x}|}{\sum f_i}$$

Where $\Sigma \rightarrow$ Represents summation

$f_i \rightarrow$ frequency

$$\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$$

Also Coefficient of Mean Deviation about mean is = $\frac{\text{Mean deviation about mean}}{\text{mean}}$

2. Standard Deviation

It is the Root Mean Square value of deviations and also called in short form as R.M.S. value.

(a) For Raw data

$$\text{Standard Deviation} = \sqrt{\frac{\sum (x_i - \bar{x})^2}{N}}$$

$\Sigma \rightarrow$ Represent Summation

where $\bar{x} = \frac{\sum x_i}{N}$

$x_i =$ value of individual data

$N \rightarrow$ Number of observations

(b) For Ungrouped and Grouped data

$$\text{Standard Deviation} = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}}$$

$\Sigma \rightarrow$ Represent Summation

where $\bar{x} = \frac{\sum f_i x_i}{\sum f_i}$

Some formulae :

$$(i) \text{ Co-efficient Standard Deviation} = \frac{\text{Standard deviation}}{\bar{x}}$$

$$(ii) \text{ Variance} = \text{Square of Standard Deviation}$$

$$(iii) \text{ Co-efficient of Variation} = \frac{S.D}{\bar{x}} \times 100$$

Example 14. Calculate mean deviation about mean and its coefficient from the following data

21, 23, 25, 28, 30, 32, 46, 38, 48, 46

Sol:

x_i	$x_i - \bar{x}$	$ x_i - \bar{x} $
21	-12.7	12.7
23	-10.7	10.7
25	-8.7	8.7
28	-5.7	5.7
30	-3.7	3.7
32	-1.7	1.7
46	12.3	12.3
38	4.3	4.3
48	14.3	14.3
46	12.3	12.3
$\Sigma x_i = 337$		$\Sigma x_i - \bar{x} = 86.4$

$$\bar{x} = \frac{\Sigma x_i}{N} = \frac{337}{10} = 33.7$$

$$M.D. = \frac{\Sigma |x_i - \bar{x}| f_i}{\Sigma f_i} = \frac{86.4}{10} = 8.64$$

$$\text{Coefficient of M.D.} = \frac{M.D.}{\bar{x}} = \frac{8.64}{33.7} = 0.26$$

Example 15. Find mean deviation about mean of the following data

x_i	:	3	5	7	9	11	13
f_i	:	2	7	10	9	5	2

Sol :

x_i	f_i	$x_i f_i$	$x_i - \bar{x}$	$ x_i - \bar{x} $	$ x_i - \bar{x} f_i$
3	2	6	-4.8	4.8	9.6
5	7	35	-2.8	2.8	19.6
7	10	70	-0.8	0.8	8.0
9	9	81	1.2	1.2	10.8
11	5	55	3.2	3.2	16.0
13	2	26	5.2	5.2	10.4
$\Sigma f_i = 35$		$\Sigma x_i f_i = 273$	$\Sigma x_i - \bar{x} f_i = 74.4$		

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{273}{35} = 7.8$$

$$M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{74.4}{35} = 2.13$$

Example 16. Find mean deviation about the mean for the following data

Marks obtained	10-20	20-30	30-40	40-50	50-60	60-70	70-80
Number of students	2	3	8	14	8	3	2

Sol :

Marks obtained	No. of students(f_i)	Mid point(x_i)	$f_i x_i$	$x_i - \bar{x}$	$ x_i - \bar{x} $	$ x_i - \bar{x} f_i$
10-20	2	15	30	-30	30	60
20-30	3	25	75	-20	20	60
30-40	8	35	280	-10	10	80
40-50	14	45	630	0	0	0
50-60	8	55	440	10	10	80
60-70	3	65	195	20	20	60
70-80	2	75	150	30	30	60
$\Sigma f_i = 40$		$\Sigma x_i f_i = 1800$		$\Sigma x_i - \bar{x} f_i = 400$		

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{1800}{40} = 45$$

$$M.D. = \frac{\sum |x_i - \bar{x}| f_i}{\sum f_i} = \frac{400}{40} = 10$$

Example 17. Calculate deviation and variance of the following data :

6, 8, 10, 12, 14, 16, 18, 20, 22, 24

Sol :

x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
6	-9	81
8	-7	49
10	-5	25
12	-3	9
14	-1	1
16	1	1
18	2	4
20	5	25
22	7	49
24	9	81
$\Sigma x_i = 150$		$\Sigma (x_i - \bar{x})^2 = 330$

$$\bar{x} = \frac{\sum x_i}{n} = \frac{150}{10} = 15$$

$$S.D. = \sqrt{\frac{\sum (x_i - \bar{x})^2}{n}} = \sqrt{\frac{330}{10}} = \sqrt{33}$$

$$\text{Variance} = (S.D.)^2 = (\sqrt{33})^2 = 33$$

Example 18. Find variance and co-efficient of variation for the following data

x_i	4	8	11	17	20	24	32
f_i	3	5	9	5	4	3	1

Sol :

x_i	f_i	$f_i x_i$	$x_i - \bar{x}$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
4	3	12	-10	100	300
8	5	40	-6	36	180
11	9	99	-3	9	81
17	5	85	3	9	45
20	4	80	6	36	144
24	3	72	10	100	300
32	1	32	18	324	324
$\Sigma f_i = 30$		$\Sigma x_i f_i = 420$			$\Sigma f_i (x_i - \bar{x})^2 = 1374$

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{420}{30} = 14$$

$$S.D = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} = \sqrt{\frac{1374}{30}} = \sqrt{45.8} = 6.77$$

$$\text{Variance} = (S.D.)^2 = 45.8$$

$$\text{Coefficient of variation} = \frac{S.D}{\bar{x}} \times 100 = \frac{6.77}{14} \times 100 = 48.35$$

Example 19. Calculate the mean, variance and coeff. of S.D. for the following distribution:

Class	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

Sol :

Class	f_i	x_i	$f_i x_i$	$(x_i - \bar{x})^2$	$f_i (x_i - \bar{x})^2$
30-40	3	35	105	729	2187
40-50	7	45	315	289	2023

50-60	12	55	660	49	588
60-70	15	65	975	9	135
70-80	8	75	600	169	1352
80-90	3	85	255	529	1587
90-100	2	95	190	1089	2178
	50		3100		10050

$$\bar{x} = \frac{\sum x_i f_i}{\sum f_i} = \frac{3100}{50} = 62$$

$$\text{Variance} = \frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i} = \frac{10050}{50} = 201$$

$$\text{S.D} = \sqrt{201} = 14.18$$

$$\text{Coff. of S.D} = \frac{\text{S.D}}{\bar{x}} = \frac{14.18}{62} = 0.22$$

EXERCISE – I

- Mean of first 10 natural numbers is
 - 5.5
 - 5
 - 4
 - 10
- Median value of first 20 natural numbers is
 - 10
 - 10.5
 - 11
 - 12
- Mode of following data : 2,2,4,4,4,5,6,7,10,11 is
 - 2
 - 4
 - 10
 - 11
- Calculate the mean of 1, 2, 3, 4, 5, 6
- Calculate the median of the data 13, 14, 16, 18, 20, 22
- Median of the following observations 68, 87, 41, 58, 77, 35, 90, 55, 92, 33 is 58. If 92 is replaced by 99 and 41 by 43, then find the new median.
- Find the mode of the set of values

2.5, 2.3, 2.2, 2.4, 2.2, 2.7, 2.7, 2.5, 2.3, 2.3, 2.6
- Find median of the series 3, 6, 6, 9, 12, 10

2. Find median of the series 4, 8, 6, 12, 15

3. Find A.M. for the following

x	:	10	11	12	13	14	15
f	:	2	6	8	6	2	6

4. Find the value of median for the following data

x	5	6	7	8	9	10	11	12	13	15	18	20
f	1	5	11	14	16	13	10	70	4	1	1	1

5. Find the mean deviation about mean of daily wages (in Rs.) of 10 workers : 13, 16, 15, 15, 18, 15, 14, 18, 16, 10 also find its co-efficient.

6. Find M.D. about mean of the following data

x	:	3	5	7	9	11	13
f	:	2	7	10	9	5	2

7. Find the mean deviation about the mean for the following data.

Income per day	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800
Number of Persons	4	8	9	10	7	5	4	3

8. Find mean deviation about mean for the following data

x_i	5	10	15	20	25
f_i	7	4	6	3	5

9. Find mean deviation for 4, 7, 8, 9, 10, 12, 13, 17

10. Find standard deviation for the following data

x_i	3	8	13	18	23
f_i	7	10	15	10	6

11. Calculate mean, variance and standard deviation for the following data

Cass Interval	30-40	40-50	50-60	60-70	70-80	80-90	90-100
Frequency	3	7	12	15	8	3	2

12. Find mean and variance for 6, 7, 10, 12, 13, 4, 8, 12

13. Find variance for the following frequency distribution

Cass Interval	0-30	30-60	60-90	90-120	120-150	150-180	180-210
Frequency	2	3	5	10	3	5	2

14. Find mean for the following data

Class interval	0-10	10-20	20-30	30-40	40-50
Frequency	5	8	15	16	6

ANSWERS

(1) a

(2) b

(3) b

(4) $7/2$

(5) 17

(6) 58

(7) 2.2

(8) 7.5

(9) 8

(10) 12.6

(11) 12

(12) 1.6

(13) 2.13

(14) 157.92

(15) 6.32

(16) 3

(17) 6.12

(18) Mean is 62, Variance is 201 and Standard Deviation is 14.18

(19) Mean is 9 and Variance is 9.25

(20) 2276

(21) 64

Software

5.3 Scilab Software-Theoretical Introduction

Scilab:

Scilab is open-source software that is used for data analysis and computation. It is also an alternative for MATLAB as this is not open-source. Scilab is named as Scientific Laboratory which resolves the problem related to numeric data and scientific visualization. It is capable of interactive calculations as well as automation of computations through programming. It provides all basic operations on matrices through built-in functions so that the trouble of developing and testing code for basic operations are completely avoided. Further, the numerous toolboxes that are available for various specialized applications make it an important tool for research. Being compatible with Matlab, all available Matlab M-files can be directly used in Scilab with the help of the Matlab to Scilab translator. The greatest features of Scilab are that it is multiplatform and is free. It is available for many operating systems including Windows, Linux and MacOS X. Some basic features of Scilab are given below:

1. It is capable to solve different algebraic equations.
2. It supports the development of certain complicated algorithms.
3. Capable of the model the previous computations.
4. Performs visualization in Bar Graphs, lines, Histograms, MathML annotation.

When we start up Scilab, we see a window shown in Fig. 5.1.

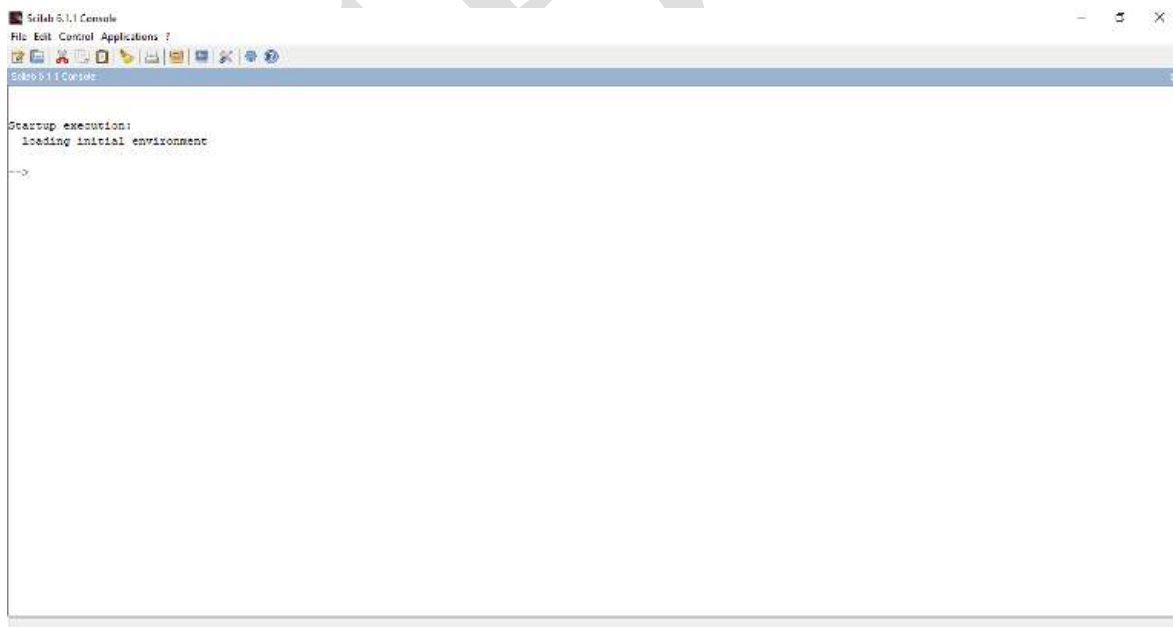


Fig. 5.1

The user enters Scilab commands after the prompt -->. But many of the commands are also available through the menu at the top. The most important menu for a beginner is the “Help” menu. Clicking on the “Help” menu opens up the *Help Browser*, showing a list of topics on which help is available (see the Fig. 5.2).

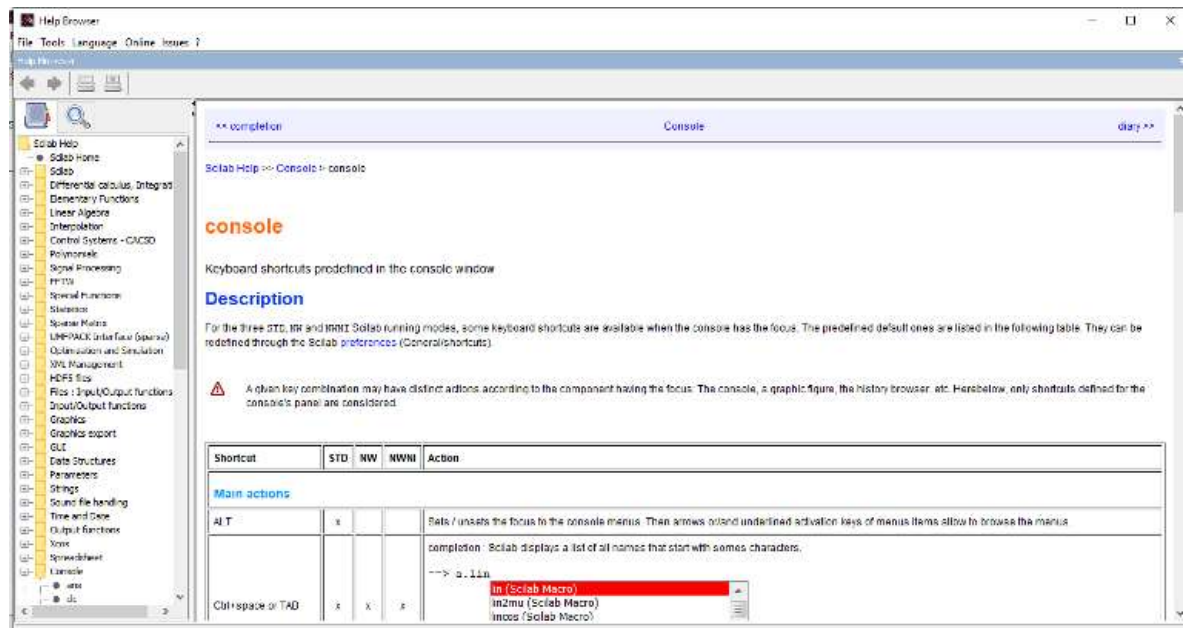


Fig. 5.2

Clicking on the relevant topic takes you to hyperlinked documents similar to web pages. The Help Browser has two tabs – *Table of Contents* and *Search*.

Table of Contents contains an alphabetically arranged list of topics and we may use the Search tab to search the help for particular topic by typing the topic in it.

Starting with Scilab

After completion of installation process, we can enter in Scilab by double-clicking on the Scilab shortcut *icon* available on our desktop. When we start Scilab, a special window called the Scilab Console appears which contains some *other* windows. The major tools within the Scilab Console are:

Scilab Console

Variable Browser

Scilab Demonstrations

The Help Browser

The Command History Window

File Browser

Graphic Window

Quitting Scilab

In order to quit Scilab, type **quit** or **exit** after the prompt, followed by pressing the enter or return key.

Entering Commands

To execute commands, every command has to be followed by enter key. Scilab commands are case sensitive and **lower case** letters are used throughout.

The Semicolon Symbol (;)

If the semicolon symbol (;) is typed at the end of a command, the output of the command is not displayed.

The clc Command

Typing **clc** command and pressing enter key cleans the command window. Once the clc command is executed, a clear window is displayed.

The clear Command

The **clear** command remove all the variables from the memory.

Help

To obtain help on a particular topic in the Scilab-list of built-in functions, e.g., Determinant, type help det after prompt.

Special Variable Names and Constants

Constant	Meaning
%pi	$\pi = 3.14159 \dots$
%e	$e = 2.71828 \dots$
%i	iota i.e. $\sqrt{-1}$
%eps	Epsilon
%inf	Infinity i.e. ∞
%nan	Not a number

Arithmetic Operations

Name of Arithmetic Operation	Symbol	Exapmle
Addition	+	$10+5 = 15$
Subtraction	-	$10-5 = 5$
Multiplication	*	$10*5 = 50$
Right Division	/	$10/5 = 2$
Left Division	\	$10\backslash 5 = 5/10 = \frac{1}{2}$

Exponentiation	$\wedge$	$10^5 = 10^5 = 100000$
----------------	----------	------------------------

Some General Commands

Command Name	Description
Clc	It clears the command window.
Clear	It clears the Workspace, all variables are removed.
clear all	Same as the command clear.
clear a b c	It clears only the variables a, b and c from the Workspace.
Clf	It clears the figure window.
Who	Lists variables currently in the Workspace.
whos	Lists variables currently in the Workspace with their sizes.

5.4 Basic Difference Between MATLAB and Scilab Software

S. No.	MATLAB	Scilab
1	MATLAB is used for performing mathematical computing.	Scilab is used for performing scientific computations.
2	MATLAB is written in C, C++ and Java.	Scilab is programmed with C, C++ and Fortran.
3	MATLAB is used for Matrix Laboratory.	Scilab is used for Scientific Laboratory.
4	MATLAB files save with extension ".m"	Scilab files save with extension ".sci"
5	The comment line begins with %	The comment line begins with //
6	It is not an open-source language.	It is open-source software.
7	MATLAB is used for solving high level computations.	Scilab is used for solving low level scientific computations.
8	The syntax for declaring an empty matrix is [] in MATLAB.	The syntax for declaring an empty matrix is [] + 1 in Scilab and returns 1.
9	Predefined variables are not write protected in MATLAB.	Predefined variables usually have the % prefix in Scilab. They are write protected.
10	Boolean variables are 0 and 1 in MATLAB.	Boolean variables are %T and %F in Scilab.
11	Polynomials are considered as vectors of coefficients in	Polynomials and Polynomial matrices are defined by the function poly in

	MATLAB.	Scilab.
12	MATLAB functions are collectively defined by (M-files)	Scilab doesn't prefer functions to load automatically; instead, it executes the command <code>getf(" ")</code> before loading it.
13	MATLAB could be run from Python Interpreter.	Scilab couldn't be run from Python Interpreter. Interactive visualizations and browser based techniques are done in Scilab.
14	MATLAB could easily learn by using Simplistic interface and meets the need of the business compared with Scilab.	Scilab is multiplatform.
15	Modulo is represented as <code>mod(a,n)</code> in MATLAB.	Modulo is represented as <code>Pmodulo(a,n)</code> in Scilab.
16	Due to paid software, MATLAB is less accessible.	Scilab is free and has less support for add-on modules.

5.5 Calculation with MATLAB or Scilab

(a) Representation of matrix (2×2 order): Here we will learn to create a matrix in MATLAB and Scilab. Some examples of matrices and their notations in MATLAB and Scilab are as follow:

Matrix Type	Type the command after Prompt (>>) in MATLAB	MATLAB Notations and Results
Row Vector or Row Matrix	<code>A=[1 0 -1 2]</code>	<pre>>> A=[1 0 -1 2] A = 1 0 -1 2</pre>

Column Vector or Column Matrix	$B = [1; 2; 3; 4]$ Or $B = [1 \ 2 \ 3 \ 4]'$	<pre>>> B=[1; 0; -1; 2]</pre> $B = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$ <pre>>> B=[1 0 -1 2]'</pre> $B = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$
Matrix of order 2×2	$A = [0 \ 1; -2 \ 3]$ Or $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$	<pre>>> A=[0 1; -2 3]</pre> $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ <pre>>> A=[0 1 -2 3]</pre> $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$

- Note:** (i) Semicolon or Enter Key is used for new row while assigning or entering a matrix.
(ii) Using the symbol ' after closing of Matrix means Transpose of the matrix.

(b) Addition, Subtraction of matrices (2×2 order) in MATLAB or SciLab: Performing addition and subtraction operation of matrices in MATLAB is so easy and interesting for beginners. Now we will learn addition and subtraction of Matrices in MATLAB by various examples in table as follow:

Assignments of Matrices	Name of Matrix Operations	Symbols of operations	Remarks

<pre>>> A=[2 5; 7 9] A = 2 5 7 9 >> B=[-5 8; -3 1] B = -5 8 -3 1 >> C=[6 -4] C = 6 -4</pre>	Assignment	=	Created three matrices A, B and C of sizes 2×2, 2×2 and 1×2 respectively by assignment operation.
<pre>>> D=A+B D = -3 13 4 10</pre>	Addition	+	Addition of matrices A and B is assigned to new variable D.
<pre>>> E=A-B E = 7 -3 10 8</pre>	Subtraction	-	Subtraction A-B is assigned to the new variable E.
<pre>>> Z=A+C Error using + Matrix dimensions must agree. >> H=B-C Error using - Matrix dimensions must agree.</pre>	Addition Subtraction	+ -	Matrices A and C are tried to be added and C is tried to subtracted from B, but MATLAB showed an error message due to their different sizes.

Note: While performing addition and subtraction of matrices, the size/order of matrices must be same, otherwise MATLAB will give an error message.

EXERCISE – II

- SciLab is a _____.
 - Software
 - Operating system
 - Hardware
 - None of these
- In SciLab, π is represented by _____.

- a. pi% b. pi c. p%i d. %pi
3. In MATLAB, π is represented by _____.
a. pi b. pii c. %pi d. None of these
4. In MATLAB, the comment line is start with _____.
5. In SciLab, the comment line is start with _____.
6. SciLab is an open-source software. (True/ False)
7. MATLAB is used for Matrix Laboratory. (True/ False)
8. SciLab is not used for Scientific Laboratory. (True/ False)
9. Write any eight differences between MATLAB and SciLab.
10. How to assign the matrix $\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$ to the variable A in MATLAB?

ANSWERS

- (1) a (2) d (3) a
(4) % (5) // (6) True
(7) True (8) False
(10) >> A=[2; 3; 0]